A MODIFICATION TO GENERAL RELATIVITY BY USE OF THE NOTION OF LOCAL EXPANSION OF SPACE-TIME

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ABSTRACT

General Relativity (GR) is Einsteins' theory of gravity whereby the motions of bodies are explained due to their following geodesic paths (the shortest distance between two points) in a curved four dimensional space-time. The curvature being attributed to the presence of mass

As such there is no explicit force present in this theory, contrary to Newtons' law of gravity. However in the limit of GR (where the field is both weak and static and bodies are travelling with velocities slow compared to that of light), then Newtons' law is recovered in the approximation.

GR has explained all gravitational phenomena extremely well (eg solar system), that is until one considers the measured motions of galaxies. The galaxies are rotating too quickly to be consistent with GR/ Newton.

An explanation for this discrepancy has originally been suggested, concerning the presence of extra mass, of as yet an unknown nature. However, to date not a hint of this 'Dark Matter' has been detected.

A second explanation is that of Modified Newtonian Dynamics (MOND), whereby at a certain very small acceleration, the actual gravitational physics deviates from that described by Newton. The MOND proposal seems to fit extremely well with observations without the need to invoke the presence of DM.

The main problem with the MOND concept is that it does not fit in with the standard GR.

The following work details an exploration into the novel assumption that the presence of mass is not only responsible for the curvature of space-time but also for a local expansion of space-time.

By introducing an expansion factor at the very beginning of standard GR analysis, one finds an extra term appears which is consistent throughout. With the inclusion of this extra term, one can, in the weak, static and slow velocity limit, find a direct link to the MOND phenomenology. It is found that the extra term is negligible for small systems (eg Solar) yet it is the dominant term for large systems (eg galaxies).

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This work is dedicated to my brother Ron Hodgkinson
6 Jan 1972 - 5 July 2003

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Chapter 1

OVERVIEW

The following work culminates in an novel attempt to modify Einstein's General Theory of Relativity (GTR) in order for the new theory to explain the dynamical behaviour of galactic systems as well as solar systems. The novel approach is to assume a *local expansion of space-time* due to the presence of mass which is then included into the standard GTR analysis (as seen in chapter seven).

The mathematics necessary for GTR, and therefore subsequently the new modified theory, is that of tensor calculus.

As such this work is set out whereby in chapter two there is a brief outline of the concepts and historical events leading to the present reason for the possible necessity of a modified GTR.

In chapter three there is then an outline of the mathematics necessary, namely the tensor calculus.

Chapter Four then introduces Einstein's Special Theory of Relativity and explains why the use of tensors is advantageous within this theory.

In Chapter Five GTR is introduced with direct links back to Chapter Three, since one is dealing with only a slight mathematical alteration of this chapter together with a physical interpretation.

Chapter Six explains the problems regarding the observed motions of galaxies.

Chapter Seven is then the tensor analysis of the new theory attempting to solve the problems encountered in Chapter Six. This modification of the standard GTR is based on the simple physical assumption that the presence of mass not only causes a curvature of space-time but also is responsible for a local expansion of it.

The concept of a tensor(any physical or geometrical object that transforms in a certain definite way) has its origins in the development of differential geometry [18].

Gauss's theory of two dimensional surfaces was extended to include any number of dimensions by Riemann [18].

The invariant differential operations for tensor analysis(tensor calculus or absolute differential calculus) were developed, as a systematic branch of mathematics, by Ricci, Levi-Civita and Weyl around the turn of the last century [18] [27]. The method of tensor calculus allows one to present all physical equations in a form independent of the choice of reference frame and as such the one can claim that it is the only possible means of studying conditions of the world which are at the basis of physical phenomena, and as such the calculus of tensors is the natural language of mathematical physics.

The introduction by Minkowski of mathematical space-time to Einstein's Theory of Special Relativity [31] [21] [1] made the above observations even more apparent, and indeed as Einstein developed and formulated his General Theory of Relativity [21] he found the calculus of tensors invaluable, in fact, one could argue, an absolute necessity.

The general theory of relativity [21] [17] [30] [19] [25] has applications in the study of planetary and local stellar motions, the study of the motion of electromagnetic radiation, the study of the entities named "black holes", the study of time itself, as well as attempts to describe the topology of the universe. However, unless a postulated, as yet unknown form of mass, can be detected, then GTR fails to predict the observed motions of galaxies and therefore possibly needs modifying.

Chapter 2

INTRODUCTION

The art of measuring the earth (geodesy) is the source of the doctrine of space, and derived its name from 'geometry' which is the Greek word for earth. With the regular change of day and night, the phases of the moon and the seasons, arose also the measure of time. These phenomena together with the attention to the stars, gave birth to the doctrine of the Universe, Cosmology.

The axioms of geometry together with the study of the motions of bodies on earth and of the 'heavens' by Ptolemy, Copernicus, Kepler, and Galilei, led to Newton's laws motion and his law of gravitation [1] [29], which was a simple inverse square law which stated that bodies of matter were attracted to one another by means of a gravitational force which was proportional to the product of their masses and inversely proportional to their separation. This approach eventually culminates in Einstein's theories of first, Special, and then General Relativity.

Galilei had realised during his work that there was a principle of relativity [21] [55] [9] regarding the laws of mechanics (the study of the interactions between matter and the forces acting on it) in any inertial reference frame (one in which bodies are free from acceleration). What this principle essentially states is that the laws of mechanics are the same for any inertial reference frame no matter what the constant velocity of that frame happens to be, and therefore, in a closed system it is impossible to determine whether you are moving with a constant velocity or are totally at rest. However this principle deals solely with mechanics and does not encompass optics (the study of electromagnetic radiation, 'light').

The wave theory of 'light' [9], which had been put on a firm foundation by James Clark Maxwell, suggested the existence of a medium (to 'carry' the waves) that must permeate all of space, and this postulated medium was given the name 'luminiferous ether'.This ether was assumed to be at rest in space.

The search for the ether culminated in the Michelson-Morley experiment [31] [55] which was an attempt to measure the difference in the speed of light as the earth travels parallel and then perpendicular to the ether drift velocity. If the ether was detected it would provide the fixed frame of reference/background for Newton's laws.

The results of the experiment failed to show up this ether drift and this was largely interpreted as revealing that an ether did not exist, although in an attempt to save the concept of an ether, Fitzgerald suggested a possible interaction between the ether and objects moving relative to it [55], such that the object became shorter in all dimensions parallel to the relative velocity. A factor supporting this theory and determining this change was derived by Lorentz [55]. Whichever of these two views is chosen the null result of the Michelson- Morley experiment indicates that all observers who measure the velocity of light will obtain the same result regardless of their own velocity through space. The speed of light is thus a constant.

The above consideration led Einstein to conclude that the speed of light does not depend on the motion of the observer and therefore there is no preferred reference frame for the laws of physics (contrary to Newton's laws which assume a fixed background). Einstein's concept had therefore introduced optics into Galilean relativity and since the speed of light was constant, all measurements (lengths and times) made by observers in different inertial reference frames, when compared parallel to their relative velocity would actually differ. However each set of measurements would be equally valid. The derived factor (the mathematics) relating the space and time measurements in any two inertial reference frames is the same as that derived by Lorentz. It is simply that the physical interpretation of what is happening is different from the idea put forward by Fitzgerald.

Einstein called his concept, which specifically dealt with inertial reference frames, the Special Theory of Relativity. The mathematician Minkowski realised that now space and time were inextricably related, transforms from one inertial frame to another could be reproduced geometrically, by a simple rotation of Cartesian coordinates in a four dimensional space-time [31]. The invariant quantity in this new space-time analogous to that of a length in ordinary space under such rotations (as found using Pythagoras's theorem) was that of the 'space-time interval'.

If accelerating frames were to be brought into this relativistic theory then since any free falling body near the earth is accelerating as measured in a frame at rest relative to the earth, then gravity would naturally be incorporated into such a theory. Einstein realised that there is no difference whatsoever between the laws of physics concerning a body at rest on a large mass and a body which is being uniformly accelerated in empty space(the Equivalence Principle). Essentially this principle states the equivalence of gravitational and inertial masses and as such he realised that there was no need for the idea of a force in order to describe the motion of bodies within the concept of gravity. It is simply that an inertial reference frame is one that is free falling within a gravitational field, so both accelerating and non-accelerating frames can be considered to be at rest.

When a rotating system (which is an accelerating system) is viewed from an inertial system, with coinciding origins around which a circle is drawn, then, due to the laws of special relativity, the circumference of the circle will appear to shorten (length contraction parallel to the motion) whereas the diameter will not. Clocks placed at the origin and on the circumference will also appear to run differently (time dilation).

The fact of the length contraction means that the usual relationship in Euclidean geometry between the constant pi (π) and the circumference of a circle no longer applies in the accelerated reference frame. The simple conclusion to this observation is that in the presence of a gravitational field the geometry is not 'flat'(Euclidean). In this situation the geometry is 'curved'. (consider a circle drawn on the surface of a ball).

Now that accelerating frames of reference had been brought into the picture, the theory of Special Relativity could be extended to a General Theory. Due to the curved geometry which now had to be included, together with the invariance of the whole concept, in order to formulate such a theory it was natural and necessary to consider the use of the mathematics of curvilinear co-ordinates, differential geometry and the concept of tensors. A tensor is defined as any physical or geometrical object that transforms in a certain definite way.

When dealing with purely spacial considerations, Gauss had developed a theory of two dimensional curved surfaces by means of curvilinear co-ordinates, by which an infinitesimal area on the curved surface could be considered virtually flat and a metric or invariant distance could be defined in such a region, and as such, the full geometry mapped using this metric, also made any choice of co-ordinates for the surface arbitrary. This was followed by developments of the mathematics in the form of the general calculus of tensors (a mathematical theory of invariants) mainly due to Ricci and Levi-Civita [27].

After Riemann had extended Gauss' idea to any number of dimensions [9], then by using the interval, as defined in the Special Theory of Relativity, an analogous system for curved space-time could be conceived. The use of the calculus of tensors meant this new physics could be formulated in an appropriate manner, whereby all physical laws could be expressed in such a way that they were independent of any particular chosen reference frame. It had been his close friend and mathematician Marcel Grossman whom had suggested the use of tensors which helped Einstein to formulate his physical theory using the existing mathematics [41].

What resulted was a formulated gravitational theory with no notion of a force acting. Bodies were simply following the shortest path they possibly could in a curved space-time (a geodesic). As such, whereas Newton's theory explains gravity as a force attracting two bodies together, Einstein explains gravity as bodies travelling along geodesics in a curved space-time, the cause of the curvature being attributed to the presence of the individual bodies or masses.

As Einstein formulated his 'General Theory of Relativity' he had realised that in the limit of his theory, where everything was moving slowly compared to the velocity of light and the gravitational field was both static and weak, that Newton's laws must come out in this approximation. This result was achieved within his analysis.

The tests conducted since that confirm General Relativity as a superior gravitational theory to that of Newton's consist of:-

1) Deflection of light near a massive body

2) The perihelion of Mercury

3) The Shapiro time delay

4) Gravitational redshift

5) Gravitational lensing (stars)

6) The apparent detection of black holes

Shortly after Einstein had published his General Theory of Relativity, Herman Weyl had attempted to generalise it further both mathematically and thus physically. His mathematical generalisation was to include a comparable length (as well as a directional) change when vectors were shifted along from point to point in a curved space-time. He attributed this difference to represent the physical electromagnetic field [31] [19]. The main failing of this idea was that the small ambiguities of the length comparisons were too small to be detected. Einstein also objected on the grounds that he suspected the atomic time and the proper time from relativity would differ as a result [46].

Since Newton's laws hold well in every day observations as an approximation to Einstein's law, when galaxies were first carefully studied and seen to be rotating, it was expected that when their rotational velocities were measured (and subsequently graphed as velocity rotation curves) and their mass calculated, that the physics would sit quite nicely with Newton's laws. However when all the luminous matter is accounted for, it was found that the observed galaxies were each rotating as a whole far too quickly for gravity to hold them together [58] [20] [44].

In order to try and explain this discrepancy, it was proposed that there maybe matter within the galaxies that could not as yet be detected and if this was allowed for, Newton's laws would hold. It was suggested that a small percentage of this matter was ordinary/known matter, whilst the majority was a completely new form of matter, never before detected [42]. This proposed new substance was termed 'Dark Matter' for obvious reasons and was thought to be in the form of Weakly Interacting Massive Particles 'WIMPS' [54].

The possible existence of Dark Matter is still to this day the most popular notion within the scientific community with which to explain the anomaly and much money and effort is being used to search for this mysterious and elusive substance [3].

However around fifty years after the suggestion of Dark Matter, after carefully studying the galactic motions, Milgrom came up with an alternative suggestion [37]. The concept was quite simple. Observations of the galaxies seemed to indicate that their motions followed a different physics to that of Newton's at a certain acceleration from the galactic centre (where the accelerations became very small) and hence as a whole the galaxies were rotating matching this new dynamics. Milgrom termed the new physics Modified Newtonian Dynamics (MOND) which has (for the weak accelerations involved) a $\frac{1}{r}$ dependence. The concept of MOND was also backed by an earlier physical observation, the Tully-Fisher relation [53] which links the luminosity of spiral galaxies to their rotational velocities.

Due to the rotation of a galaxy, an observer will naturally see part of the galaxy moving towards him and part of the galaxy moving away,thus light will be either blue shifted or red shifted respectively.

The spectral line of a chosen element will be smeared out due to this motion and therefore the broader the line the faster the galaxy is spinning. Since the luminousity (the total amount of energy emitted per unit time) of the galaxy is directly proportional to its baryonic mass (M), then the total luminousity is therefore proportional to the rotational velocity (v).

The Tully-Fisher relationship is found to be $M \propto v^4$ where the rotational velocity is independent of the distance.

There are several different types of galaxies (spiral, elliptical, dwarf etc) and although MOND has a very inflexible form it is seen to fit a wide range of independent galaxies very well indeed. Whereas the Dark Matter hypothesis has several free parameters that can appear with any magnitude and with any distribution for the fits. This fact was well exemplified when attempts were made to fit the rotation curves of a fake galaxy (the mass distribution of one together with the rotation curve of another). Having no free parameters MOND cannot fit the curve, whilst the Dark Matter model can easily fit the fictitious object.

The difficulty about accepting MOND as a complete theory is that there was no relativistic underpinning to it. Newton's laws are the approximation in the limit of Einstein's General Theory of Relativity. There have been attempts to modify Einstein's theory in order to underpin MOND, most notably by Beckenstein [7] and also Mannheim [24]. However these theories are not wholly satisfactory.

An outline of the mathematics necessary to describe GTR and therefore the new modification that is presented in Chapter Seven, which incorporates MOND, now follows.

Chapter 3

TRANSFORMATIONS, TENSORS AND CURVED SPACES

3.0.1 Transformations

If time and space are considered absolute entities and are totally independent of one another, as in the Newtonian description of the world [9], then it is possible to describe a physical or geometrical object in either space or time separately and relate the two as and when convenient.

Let us look at the analysis starting with the description of physical and geometrical objects in space. The subsequent analysis outlines the same arguments as laid out in [30], [21] and [17].

If in two dimensional Euclidean ('flat') space, one is to describe a straight line or vector, a defined origin and co-ordinate system are chosen.

If in two dimensional space the co-ordinate system is Cartesian then one denotes two orthogonal axes x_1 and x_2 (which are continuous functions). Then two values from the origin in both the x_1 and x_2 co-ordinates axes can be used to specify two points in the coordinate system in between which an object is located, the object being that of a straight line or interval (s). The differences (Δ) of the x_1 values and of the x_2 values (the components) can be used to specify the length of the straight line. Since moving this vector but keeping it parallel to itself will result in the same vector, this representation in Euclidean space allows the notion of a vector which is not necessarily located at a definite point, i.e. a free vector.

The length of the interval can be found using Pythagoras' theorem.

$$s^2 = \Delta x_1^2 + \Delta x_2^2 \tag{3.1}$$

Now, supposing that at the origin the co-ordinate system is rotated (see fig. 3.1) and the same interval is described using this new co-ordinate system (a type of transformation [10] [27]), the components (co-ordinate differences) will change but the interval itself will obviously not. It is covariant under the change of co-ordinates. The length of the interval is also obviously unchanged and is thus a scalar invariant.

If Equation 3.1 holds for any orientation then this is what actually defines Euclidean space and Cartesian co-ordinates.

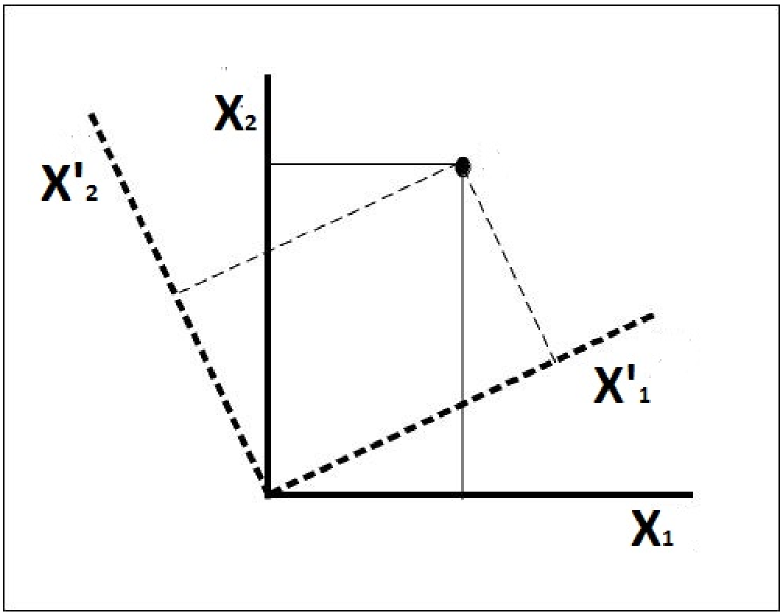

FIGURE 3.1: Rotation

The length of a definite interval can be defined to be unity and thus the length of any other interval can be determined by multiplication of this unit interval and as such a unit of measure (a metric) has been constructed which is independent of the co-ordinates. This construction will naturally follow for an infinitely small interval (ds). Also this form can be extended to any number of dimensions. The expressions are as follows.

In the case of a general rotational and translational co-ordinate transformation of Cartesian form the relationship between two co-ordinate systems, x'_m and x_m is necessarily linear and of the form,

$$x'_m = \sum_n a_{mn} x_n + b_m \tag{3.2}$$

Here the b_m is a constant and is the translational shift of the origin. The coordinate indices m and n would obviously, for three dimensional space, range from 1 to 3 (giving three equations). The a_{mn} are the rotational transformation coefficients (here they are not functions of the co-ordinates). The x'_m can also be thought of as points in the coordinate systems and so, using the notation above one can show this transformation of a vector by

$$\Delta x'_m = \sum_n a_{mn} \Delta x_n \tag{3.3}$$

On account of the scalar invariance of the length in 3.3 and the definite interval, one finds that

$$s^2 = \sum_m \Delta x_m^2 = \sum_m \Delta x_m'^2 \tag{3.4}$$

One can also express these transformations in matrix form such that equation 3.3 has the equivalent matrix form,

$$z' = \mathbf{A} z \tag{3.5}$$

Where, $z' = \Delta x'_m$ and $\mathbf{A} = a_{mn}$

Matrix algebra involving these transformations leads to,

$$\mathbf{A}\mathbf{A}^T = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{3.6}$$

Where,

$\mathbf{A}^T$ is the transpose of $\mathbf{A}$, $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$, and $\mathbf{I}$ is the unit or identity matrix.

$\mathbf{I}$ can be written in component form as

$$\mathbf{I} = \delta_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.7}$$

again here one is dealing with three dimensions.

The component expression of 3.6 is therefore

$$\sum_m a_{mn} a_{mk} = \delta_{nk} \tag{3.8}$$

The inverse transformation of 3.3 is

$$\Delta x_k = \sum_m a_{mk} \Delta x'_m \tag{3.9}$$

This shows that the same coefficients, a, also determine the inverse transformation. Geometrically a_{mn} is the cosine of the angle between the x' axis and the x axis.

With the use of the δ_{mn} the infinitesimal interval can be in differential form as

$$ds^2 = \sum_{mn} \delta_{mn} dx_m dx_n \tag{3.10}$$

As the distance s between two points is expressed using Cartesian coordinates in this very simple manner, then one can say that the Cartesian systems are the preferred systems in Euclidean geometry. The whole of geometry may be founded on this concept of distance.

A volume element is another example of an invariant quantity, the determinant is scaled to unity thus there is no change in volume with respect to a linear orthogonal transformation. It is independent of the particular choice of Cartesian coordinates and is therefore invariant.

If one denotes the coordinate differences in 3.3 by a vector, itself in Cartesian coordinates, say A, one has

$$A'_m = \sum_n a_{mn} A_n \tag{3.11}$$

With m and n now called component indices.

The above equation of a straight line/vector under this transformation, has the same form for all vectors by which the components change but the vector itself does not. Vectors therefore also have an objective significance when they are transformed in Cartesian coordinates. As stated earlier the unit interval can be used to build any vector by scalar multiplication of the unit interval.

Therefore the equation of a straight line is covariant with respect to linear orthogonal transformations.

In order to condense the writing (thus following the accepted Einstein convention) the summation sign in any expression is now omitted and it is understood that the summation in the expression is to be carried out for those indices that appear twice. The summation indices simply state a sum is to be taken and therefore they may be changed to any other symbol as and when convenient *providing the new symbol is not already denoting components within the expression.*

3.0.2 Tensors

Now, if one has two vectors, A_m and B_m and the n^2 quantities $A_m B_n$ are considered, then under a linear orthogonal transformation, this object will transform as

$$A'_m B'_n = a_{mk} a_{nl} A_k B_l \tag{3.12}$$

Or equivalently this can be written as

$$C'_{mn} = a_{mk} a_{nl} C_{kl} \tag{3.13}$$

An object such as this (which is termed a 'tensor') may not necessarily be the product of two vectors but as long as it transforms in exactly this manner it is defined as a second order/rank tensor (second because of the two indices).

If a tensor is symmetric (eg $A_{mnp} = A_{nmp}$) or antisymmetric (eg $A_{mnp} = -A_{nmp}$) between *any* pair of indices then this symmetry is preserved upon transformation.

These transformation laws can be extended to tensors of higher order (third order, etc) and it is therefore reasonable to term a vector as a first order tensor and a scalar invariant as a zeroth order tensor.

If one takes the second order tensor, δ_{mn}, as defined in 3.7, then under transformation one has

$$\delta'_{mn} = a_{mk}a_{nl}\delta_{kl} = a_{mk}a_{nk} = \delta_{mn} \tag{3.14}$$

Now since this particular tensor has components which are the same relative to all sets of Cartesian coordinates, it is termed the 'fundamental' tensor of second order.

New tensors may be formed by addition or subtraction of the corresponding components of tensors which are of the same order. The resulting tensor will be one of the same order, so

$$A_{mnp}\ldots \pm B_{mnp}\ldots = C_{mnp\ldots} \tag{3.15}$$

Tensors of different or equal order may have their components multiplied together (an outer product), the resulting tensor will be one which is the sum of the original orders, so

$$A_{mnp}\ldots B_{qrs}\ldots = T_{mnp\ldots qrs\ldots} \tag{3.16}$$

A tensor may also be 'contracted' by means of setting two of its sets of components equal to one another. It follows then that these sets of components become the summation and the resulting tensor is now two orders less, so

$$A_{mmp....} = T_{p....} \tag{3.17}$$

Tensors may also be formed through differentiation, as such

$$A_{mnp.....v} = \frac{\partial B_{mnp....}}{\partial x_v} \tag{3.18}$$

3.0.3 Curvilinear Coordinates and Curved Spaces

Now transformations need not be of the previous special orthogonal category. Transformations to spherical coordinates (etc) in which the transformation is neither orthogonal nor linear (they are termed 'curvilinear') are obviously possible.

In order to emphasise the usefulness of coordinate transforms just in a purely mathematical sense one can consider the equation of a circle of radius s in Cartesian coordinates which can be expressed as that of Pythagoras' theorem (3.1), if one transforms to polar coordinates this equation can be expressed in even simpler terms such that

$$s = \Delta x^1 \tag{3.19}$$

In fact the dramatic change that takes place in a differential equation under a change of variables is nothing but a change of coordinates.

To generalise transformations, one should start, however, by considering the transformation from Cartesian coordinates to oblique coordinates, in particular consider the transformation of the invariant quantity of the interval s. If one denotes the components of a vector in the oblique system, x^m, and the orthogonal projections of this vector onto the axis x_m, (note in the Cartesian system, there is no difference between the components of the vector and the orthogonal projections) (see fig. 3.2).

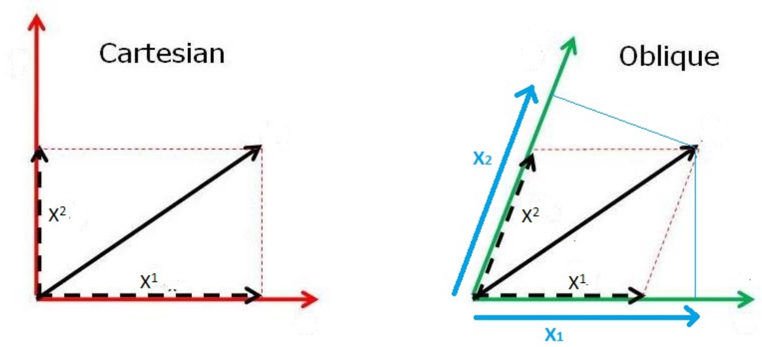

FIGURE 3.2: Orthogonal Projections

One finds that in order to preserve the length of the interval under such a transformation leads to

$$s^2 = x^m x_m \tag{3.20}$$

It follows that for an infinitesimal interval ds one has the differential form

$$ds^2 = dx^m dx_m \tag{3.21}$$

Or equivalently

$$ds^2 = g_{mn} dx^m dx^n \tag{3.22}$$

which is a generalised version of Pythagoras' Theorem.

One should note here that if the space is flat and Cartesian coordinates are chosen then the g_{mn} are equivalent to the δ_{mn} of 3.10

The dx_m can be thought of as components of a vector themselves. To distinguish, one therefore calls the dx^m 'contravariant' components and the dx_m 'covariant' components. One therefore has two different kinds of vector (components), eg A^m and A_m.

In 3.22 the g_{mn} are thus the set of coefficients (scaling factors) for the length preservation of the interval/metric under this transformation and as such effectively maps (scales) the contravariant components to the covariant components and vice versa. Therefore

$$A_m = g_{mn} A^n \tag{3.23}$$

And

$$A^n = g^{mn} A_m \tag{3.24}$$

where 3.24 is an inverse transformation. Since $g_{mn}A^mA^n$ is an invariant, it follows that the form $g_{mn}A^mB^n$ is also an invariant, which is termed the scalar/inner product.

These transformations from Cartesian coordinates to oblique coordinates, although still linear, are obviously more general than the linear orthogonal transformations. To go further and encompass all tensor transformations (a generalisation that will obviously include the afore mentioned curvilinear transformations in Euclidean/'flat' space) one can consider general curvilinear coordinates in the form of the geometry of 'curved' surfaces/spaces.

So, if one is dealing with an arbitrary curved space in two dimensions (the premise can later be extended to higher dimensions), a continuous mesh can be 'drawn' on the surface, where due to the curvature both angle and interval of intersections on the mesh will be a function of position. By considering each infinitesimal area to be 'flat' and using 3.22 for each infinitesimal interval then the whole geometry of the surface can be mapped out since the g_{mn} can scale and preserve the interval from point to point. This means that the g_{mn} will have different values from point to point and as such is now a field quantity. Since one is now dealing with curved space the indices notation is changed here to $g_{\alpha\beta}$ in order to reflect this fact. Thus the expression

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{3.25}$$

is now a full generalisation of the Pythagoras theorem (since it is now a field quantity) and since one can transform from one $g_{\alpha\beta}$ to another (shown later) this makes the choice of coordinates totally arbitrary.

The transformation coefficients of a vector (contravariant or covariant) will now be functions of the coordinates, and as such there are no free vectors in curved space.

The transformation law for a contravariant vector A is thus

$$A^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} A^{\beta} \tag{3.26}$$

where now, for clearer notation, the dash denoting a different system now appears with the component index. For a more concise expression the notation for 3.26 can be changed to

$$A^{\alpha'} = x^{\alpha'}_{,\beta} A^{\beta} \tag{3.27}$$

where the comma denotes the differentiation.

The transformation law for a covariant vector B is derived to be

$$B_{\alpha'} = x^{\beta}_{,\alpha'} B_{\beta} \tag{3.28}$$

The above transformation laws gives one rules for general tensors and a basis for a general theory of invariants.

One can show also that

$$g_{\alpha'\beta'} = g_{\mu\nu} x^{\mu}_{,\alpha'} x^{\nu}_{,\beta'} \tag{3.29}$$

so the $g_{\alpha\beta}$ is a tensor (the 'metric' tensor, which is symmetrical), and so is $g^{\alpha\beta}$. They are the fundamental tensors in curved space. The $g^{\alpha\beta}$ equals the co-factor of the corresponding $g_{\alpha\beta}$ in the determinant of the $g_{\alpha\beta}$, divided by the determinant itself, hence the $g^{\alpha\beta}$ is also symmetric.

From 3.23 and 3.24 one also has the expression

$$g_{\alpha\beta} g^{\beta\gamma} = g^{\gamma}_{\alpha} \tag{3.30}$$

where $g^{\gamma}_{\alpha} = 1$ for $\gamma = \alpha$ and $g^{\gamma}_{\alpha} = 0$ otherwise.

The volume element is found to be invariant by the use of $\sqrt{g'}dx' = \sqrt{g}dx$ where g is the determinant of $g_{\alpha\beta}$.

Now suppose S is a scalar field quantity, and as such is therefore a function of either x^{α} or $x^{\alpha'}$, then by partial differentiation one has

$$S_{,\alpha'} = S_{,\rho} x^{\rho}_{,\alpha'} \tag{3.31}$$

and thus the derivative of a scalar field is a covariant vector.

3.0.4 Parallel Transport and the Christoffel Symbols

As stated earlier, in curved space, the metric tensor is a field quantity and therefore there are no free vectors, so attempting to parallel transport a vector from one point to another runs into difficulties. However if one examines a point very close to the original point, P (where a vector is present) one can find a vector parallel to the original one with an uncertainty of the second order. A vector can be transported along a path in this manner to a further point Q and as such give some sort of meaning to the notion of parallel displacement in curved space, by keeping the length constant, and keeping track of the uncertainty, although different paths from P to Q would give different results for the final vector. Further a vector transferred/transported along a closed loop circuit in this manner would result, in general, in a different vector to the original.

In order to get equations for this parallel displacement, the simplest method is for one to imagine the curved space embedded in a flat (or tangent) space of higher dimensions. The use of this flat higher dimensional space, in this instance, is purely to obtain equations for the parallel displacement of a vector. By introducing this flat higher N dimensional space one can choose rectilinear co-ordinates z^n $(n = 1, 2, 3, ...N)$. The co-ordinates do not need to be orthogonal, only rectilinear.

Since each co-ordinate y^n is a function of the four x co-ordinates, $y^n(x)$, then the equations of the surface can be found by eliminating the four $x's$ from the $Ny^n(x)'s$ and as such there are $N - 4$ such equations.

By differentiating the $y^n(x)$ with respect to the parameters x^μ one has

$$\frac{\partial y^n(x)}{\partial x^\mu} = y^n_{,\mu}$$

In this flat space the invariant distance between two neighbouring points is

$$ds^2 = h_{mn} dx^m dx^n \tag{3.32}$$

Where the h_{mn} are obviously constants due to the space being flat.

Now each point, x^α, in the curved space determines a definite corresponding point y^n in the flat higher dimensional space.

So for two neighbouring points in the surface differing by δx^α one has

$$\delta y^n = y^n_{,\alpha} \delta x^\alpha \tag{3.33}$$

By use of the equations of the squared distances in both the flat higher dimensional space and the curved space and the fact that the h_{mn} are constants, one can find the relationship

$$g_{\alpha\beta} = y^n_{,\alpha} y_{n,\beta} \tag{3.34}$$

Now in 3.33 the δy^n can itself be considered a vector and so accordingly 3.33 can be rewritten as

$$A^n = y^n_{,\alpha} A^\alpha \tag{3.35}$$

where A^n is the contravariant vector in the higher dimensional flat space and A^α is the contravariant vector in the curved space.

Now under ordinary parallel displacement $(x + dx)$ this vector will no longer lie in the surface of the curved space but it can be projected back onto the surface. The method is to split the vector into a tangential part and a normal part and then quite simply discard the normal part since the normal part is defined to be tangential to every tangential vector at the point $x + dx$.

The change in the vector (dA_α)under this parallel displacement is found to be

$$dA_\beta = A^\alpha y^n_{,\alpha} y_{n,\beta,\gamma} dx^\gamma \tag{3.36}$$

Now by differentiation and careful algebraic manipulation of 3.34 one can find the relationship

$$y^n_{,\alpha} y_{n,\beta,\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}) \tag{3.37}$$

Let

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}) \tag{3.38}$$

Note that the object $\Gamma_{\alpha\beta\gamma}$ (known as the Christoffel symbols) does not itself transform as a tensor. However the indices may be raised or lowered (the mapping of the contravariant components to the covariant components and vice versa) in the usual fashion by use of the metric tensor. It is symmetric between the last two indices.

A useful expression, stemming directly from 3.38, is

$$\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = g_{\alpha\beta,\gamma} \tag{3.39}$$

So now using 3.38 the change in the vector under parallel displacement can be expressed without any mention of the higher dimensional space and, as such, one is dealing with the curved space alone.

The change under parallel displacement for a covariant vector is now

$$dA_\beta = \Gamma^\alpha_{\beta\gamma} A_\alpha dx^\gamma \tag{3.40}$$

and the change for a contravariant vector is

$$dB^\beta = -\Gamma^\beta_{\alpha\gamma} B^\alpha dx^\gamma \tag{3.41}$$

The length of the vector does remain constant under this transport since the normal part which has been discarded is infinitesimal so that to the first order the length of the whole part equals that of the tangential part.

3.0.5 Geodesics

Now it is possible to use this method of parallel displacement for curved spaces in order to determine the shortest distance between two points (a geodesic) on such a surface.

The method is to take a point with coordinates say z^α and suppose that it moves along a path of some parameter τ. The tangent vector to the path is then

$$\frac{dz^\alpha}{d\tau} = u^\alpha \tag{3.42}$$

Now if this vector (which can be scaled to unity) is moved under parallel displacement along the path by moving the initial point to the point $z^\alpha + u^\alpha d^\tau$ and then moving the vector u^α to this new point, then if this process is repeated (however many times) the result is a geodesic line.

So by substituting $B^\beta = u^\beta$ together with $dx^\gamma = dz^\gamma$ into 3.41 and then using 3.42 one can acquire the equation for a geodesic, which is

$$\frac{d^2 z^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\gamma} \frac{dz^\alpha}{d\tau} \frac{dz^\gamma}{d\tau} = 0 \tag{3.43}$$

The geodesic line is the invariant form in curved space analogous to the straight line in Euclidean space.

It should be noted that the same form for a geodesic line may be obtained by finding the stationary value to the integral $\int ds$ between two points [17].

3.0.6 Covariant Differentiation

Now if one wants to define the differentiation of a vector field ($A_{\alpha,\beta}$) one needs to compare the values of two neighbouring but distinct points in the vector field, but since the $g_{\alpha\beta}$ are field quantities in curved spaces and as such vary from point to point, such a comparison cannot lead to a tensor (unless one is dealing with a linear transformation) and therefore this operation does not conform with a general theory of invariants.

The problem can be seen below, where the presence of the last term on the right hand side reveals that $A_{\alpha,\beta}$ does not transform as a tensor. With use of 3.28 one gets

$$A_{\alpha',\beta'} = (A_\gamma x^\gamma_{,\alpha'})_{,\beta'} = A_{\gamma,\sigma} x^\sigma_{,\beta'} x^\gamma_{,\alpha'} + A_\gamma x^\gamma_{,\alpha',\beta'} \tag{3.44}$$

One can however modify the process of differentiation in order to obtain a tensor. If one takes a vector say A_α at a point x and moves it through parallel transport to a new point $x + dx$, it may be thought of as the same vector defined at the neighbouring point. However as seen earlier due to the curvature the components will be different and this change is expressed in 3.40. One can now take the difference between the displaced vector and the original vector which is now defined at the new point. The result is a vector for the small displacement $dx_{,\beta}$. The subtraction of the two vectors yields

$$A_\alpha(x+dx) - [A_\alpha(x) + \Gamma^\gamma_{\alpha\beta} A_\gamma dx^\beta] \tag{3.45}$$

When taken to the first order this is found to equal

$$(A_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta} A_\gamma) dx^\beta \tag{3.46}$$

The coefficient of dx^β is itself a general tensor which of course now satisfies the mathematical concept of general covariance and so one calls it the covariant derivative of A_α. This modified differentiation is expressed as

$$A_{\alpha:\beta} = A_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta} A_\gamma \tag{3.47}$$

Where the : denotes the covariant differentiation.

The covariant derivative of the outer product $X_\alpha Y_\beta$ is now defined to be

$$(X_\alpha Y_\beta)_{:\sigma} = XY_{:\sigma} + YX_{:\sigma} = (X_\alpha Y_\beta)_{,\sigma} - X_\alpha Y_\gamma \Gamma^\gamma_{\beta\sigma} - X_\gamma Y_\beta \Gamma^\gamma_{\alpha\sigma} \tag{3.48}$$

Now, as seen in 3.16, the outer product of two or more tensors may be expressed as a single tensor, so the covariant derivative of a second order tensor, for example, will be

$$T_{\alpha\beta:\sigma} = T_{\alpha\beta,\sigma} - T_{\alpha\gamma}\Gamma^\gamma_{\beta\sigma} - T_{\gamma\beta}\Gamma^\gamma_{\alpha\sigma} \tag{3.49}$$

This process may be extended for tensors of higher orders with an extra term appearing for each additional set of components.

The covariant derivative of a scalar quantity is just the same as the ordinary derivative, so one has

$$S_{:\sigma} = S_{,\sigma} \tag{3.50}$$

Following the same logic as for the covaviant derivative of an outer product and with use of 3.47 and 3.50, one gets for the covariant derivative of the scalar product $(A^{\alpha}B_{\alpha})_{:\sigma}$ the result

$$(A^{\alpha}B_{\alpha})_{,\sigma} = A^{\alpha}(B_{\alpha,\sigma} - \Gamma^{\gamma}_{\alpha\sigma}B_{\gamma}) + B_{\alpha}A^{\alpha}_{:\sigma} \tag{3.51}$$

This then leads to

$$A^{\alpha}_{:\sigma} = A^{\alpha}_{,\sigma} + \Gamma^{\alpha}_{\gamma\sigma}A^{\gamma} \tag{3.52}$$

So 3.52 is now the definition of the covariant derivative of a contravariant vector, which can be extended in a similar fashion as for a covariant vector to contravariant tensors of higher order.

It should also be noted that from 3.49 and the relationship $\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\gamma\alpha} = g_{\alpha\beta,\gamma}$ one can get the result

$$g_{\alpha\beta:\sigma} = 0 \tag{3.53}$$

This shows that the components of the metric tensor are constants under covariant differentiation and as such the indices of tensors may be raised or lowered before the operation of covariant differentiation is performed and the result is the same as if they are moved in this way afterwards.

3.0.7 Riemannian Curvature

Now, the operation of covariant differentiation has a fundamental difference from that of ordinary differentiation when two operations are performed in succession and the order of these operations is considered. For ordinary differentiation the order does not matter. However for covariant differentiation the order does matter. The reason for this difference is simply the curvature causing the path dependent result of a vector under parallel transport and can be understood in the following way.

If one takes a point P in curved space with a vector A_α present, if one now moves this vector by parallel transport around an infinitesimal closed loop back to the original point P, as seen earlier there will, in general, be a change in the vector. The result of this process is equivalent to taking the second covariant derivatives of A_α ($A_{\alpha:\beta:\gamma}$) which can be considered a point midway around the loop, and then subtracting this result from the result of the second covariant derivatives to this midway point but in the opposite direction ($A_{\alpha:\gamma:\beta}$). The end result is not zero, showing that the order of these operations does indeed matter as seen below

$$A_{\alpha:\beta:\gamma} - A_{\alpha:\gamma:\beta} = A_\sigma R^\sigma_{\alpha\beta\gamma} \tag{3.54}$$

Where

$$R^\sigma_{\alpha\beta\gamma} = \Gamma^\sigma_{\alpha\gamma,\beta} - \Gamma^\sigma_{\alpha\beta,\gamma} + \Gamma^\tau_{\alpha\gamma}\Gamma^\sigma_{\gamma\beta} - \Gamma^\tau_{\alpha\beta}\Gamma^\sigma_{\tau\gamma} \tag{3.55}$$

This is known as the curvature tensor (Riemann-Christoffel) which gives a measure of the deviation from flat space.

It has the property of symmetries such that

$$R^\sigma_{\alpha\beta\gamma} = -R^\sigma_{\alpha\gamma\beta} \tag{3.56}$$

$$R_{\sigma\alpha\beta\gamma} = -R_{\alpha\sigma\beta\gamma} \tag{3.57}$$

$$R_{\sigma\alpha\beta\gamma} = R_{\beta\gamma\sigma\alpha} = R_{\gamma\alpha\beta\sigma} \tag{3.58}$$

Now if space is flat, one may choose a rectilinear system of coordinates. Thus the $g_{\alpha\beta}$ are constants and since the Christoffel symbols are made up of derivatives of the metric tensor, one gets as expected

$$R_{\alpha\beta\gamma\sigma} = 0 \tag{3.59}$$

Also working the other way, one can prove space is flat if the curvature tensor vanishes.

The curvature tensor can be contracted in the usual manner, so (avoiding components which are anti-symmetrical) one gets, first, the Ricci tensor

$$R^{\alpha}_{\beta\gamma\alpha} = R_{\beta\gamma} \tag{3.60}$$

which is a symmetrical tensor.

and secondly, after a further contraction one gets the Ricci scalar

$$R^{\beta}_{\beta} = R \tag{3.61}$$

There are some useful expressions involving the covariant derivatives of the curvature, Ricci and Ricci scalar tensors, the Bianci relations ;

$$R^{\alpha}_{\beta\gamma\rho:\sigma} + R^{\alpha}_{\beta\rho\sigma:\gamma} + R^{\alpha}_{\beta\sigma\gamma:\rho} = 0 \tag{3.62}$$

and

$$(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R)_{:\gamma} = 0 \tag{3.63}$$

So one now has an outline of the mathematics of transformations, tensors, and the calculus of tensors both within the properties of ordinary flat spaces and curved spaces. In the following chapter it will be seen that an analogous quantity of the invariant *interval* as discussed in this chapter can be seen to be the basis of the representation of the physics of Special Relativity, where one has a four dimensional space-time continuum.

In Chapter Five it will be seen that for Einstein's General theory of relativity (a theory of gravity) this space-time continuum will become curved and as such the mathematics (with the new interval) will follow the same form as in this chapter and will then have a real physical interpretation for the laws of nature.

Chapter 4

RELATIVITY

4.0.8 Galilean Relativity

Galilean Relativity states that under the natural physical laws of mechanics there is no way one can tell the difference, without external reference, between being at rest or travelling with uniform motion (a constant linear velocity). If one is in this state, it is classed as an *inertial reference frame.*

Now travelling with constant velocity means to cover equal distances in equal amounts of time. However on reflection of the above paragraph, it is impossible to clearly and definitely mark out equal distances and to be sure of definite time intervals.

Isaac Newton overcame this problem, and in so doing extensively developed the classical mechanics of Galileo, by introducing the concept of *absolute space* and *absolute time* [1].

Newton's notion of absolute space was such that, throughout the entirity of space there existed a kind of 'lattice' consisting of points with a definite invariant measure/distance between them (a fixed background reference frame). His notion of absolute time was that of the existence of a universal clock that ticked at a definite invariant rate.

Neither absolute space nor absolute time were observable but Newton assumed their presence in order to formulate consistent laws of motion.

It is probably worth noting here that Gottfried Leibniz argued with Newton concerning his assumption of absolute space and time. Leibniz was of the opinion that all motion was a relative concept. In fact using what he termed his *Principle of Sufficient Reason* [26], he asked the question, to paraphrase "if absolute space exists then why is everything located as it is and not, say, a few feet to the left?" His argument was similar concerning absolute time, asking "why was everything not created a year earlier?".

Leibniz's viewpoint of relative motion was later take up by Earnest Mach, who on realising that the universe itself was not rotating relative to our sense of rotation, asked the question, to paraphrase and put quite simply, if the universe was rotating would there be an immediate detectable effect in a local inertial reference frame? Leibniz and Mach would have answered "yes" to this question, since they believed all motion was relative, however Newton would have answered "no" since he would claim the universe was rotating solely with respect to absolute space. This became known as *Mach's Principle* [30]

4.0.9 Newton's Laws

However returning to Newton's concept of mechanics, using his notion of a fixed background reference frame, Newton formulated his laws of mechanics, which stated that

1) Every object persists in its state of rest or uniform motion in a straight line unless it is compelled to change that state by forces impressed on it.

2) Force (F) is equal to the change in momentum (mv), mass(m) velocity (v), per change in time. For a constant mass, force equals mass times acceleration (a).

$$F = ma \tag{4.1}$$

3) For every action, there is an equal and opposite reaction.

Newton took his laws and with them, since any object freely falling to the earth is actually accelerating, described the phenomenon of *gravity* as an attractive force between two objects, thereby explaining the motions of the solar system.

Newton's law of gravity (gravitation) stated that...

Every object in the universe attracts every other object, with a force directed along the line of centers for the two objects that is proportional to the product of their masses and inversely proportional to the square of the separation between the two objects. This law is expressed as

$$F = G\frac{m_1 m_2}{r^2} \tag{4.2}$$

and therefore, expressed as the acceleration potential Φ, one has

$$\Phi = -\frac{m}{r} \tag{4.3}$$

where, for consistency with later tensor analysis, units are chosen such that G is equal to unity.

Here G is the gravitational constant, m_1 and m_2 denote the two masses and r is the separation distance.

Now 4.3 satisfies the Poisson equation [30], which is

$$\bigtriangledown^2 \Phi = 4\pi\rho \tag{4.4}$$

where ρ is the density of matter.

Newton's law of gravity explained the motions of the planets in the solar system very well, except for the orbit of Mercury.

It should be noted here that the inertial mass of 4.1 and the gravitational mass of 4.2 turn out experimentally, for no obvious reason, to be equivalent, meaning that objects of different masses will accelerate at the same rate in the same gravitational field [1].

When considering 4.1 and accelerated reference frames, Newton introduced the concept of *inertial forces* which allows one to treat accelerated frames as inertial frames with these fictitious forces employed [29].

4.0.10 Galilean Transformations

Now one needs to see how mechanical events in one system (frame of reference/observer) moving with uniform motion appear from another system also moving with uniform motion. When one considers the principle of Galilean-Newton relativity it is obvious that no generality is lost if one system (x_m) is considered to be at rest (wrt absolute space) and the other system (x'_m) is considered to be moving with constant linear velocity (v), and time (t). Using Cartesian coordinates, with an initial coinciding origin and motion parallel the corresponding x_1 axis in both frames. Then one finds the resulting linear orthogonal transformations (as such there is no difference between covariant and contravariant components) are translational and of the form 3.2 such that [55] [9],

$$x'_1 = x_1 - vt \tag{4.5}$$

$$x'_2 = x_2 \tag{4.6}$$

$$x'_3 = x_3 \tag{4.7}$$

$$t' = t \tag{4.8}$$

If u and u' are the velocities of a test particle relative to the two frames of reference then this situation results in the velocity addition formula

$$u_1' = u_1 - v \tag{4.9}$$

From the conservation of momentum and the above transformations one can show that $m' = m$, $a' = a$ and $F' = F$ [30]. This immediately follows from the fact that distance and time are invariant quantities under these transformations. As such Newton's laws of motion are covariant with respect to transformation between inertial frames. However Newton's laws deal solely with mechanics and therefore do not encompass the phenomena of optics/electromagnetic radiation/'light'.

4.0.11 Michelson-Morley Experiment

All experiments concerning the velocity of light (c) had and have always resulted in the same value $c \simeq 3 \times 10^8$ meters per second. In addition, James Clerk Maxwell's field equations for the wave propagation of light found the same value for velocity, determined purely from constants [31].

Since Maxwell's theory showed light to be an electromagnetic wave motion, it was proposed that there must be a medium present to support the waves. This assumed stationary medium was termed the *aether* and the Michelson-Morley experiment was designed to detect it. The detection of such a medium would have been conducive to Newton's concept of absolute space [31] [55] [9].

With the view of this stationary/absolute aether, then during the relative motion of the earth through the aether (or conversely the aether drift relative to the earth) then one should be able to take measurements, using a large equal arm interferometer, of light velocity that would differ. If the beam splitter in the interferometer is angled such that one path is parallel to the emitted beam of light and the other path is normal to it, the the orientation can be such that one path s parallel to the aether drift and the other path normal to the aether drift. By simple addition and trigonometry, the velocity should be less for light travelling to a point and back in the situation normal to the aether drift than that of the light velocity to a point and back in the situation parallel to the aether drift.

The experiment found the velocity of light to be exactly the same repeatedly for both measurements. As such this null result showed the velocity of light to be a constant with respect to any inertial reference frame.

In an attempt to save the concept of the aether, Fitzgerald and Lorentz suggested that objects *actually* contracted in length parallel to the relative velocity [55]. The ad-hoc factor they derived for this length contraction was

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4.10}$$

4.0.12 Special Relativity

Einstein took the simple viewpoint such that the aether did not exist and therefore there was no preferred reference frame and that *all* laws of physics are covariant with respect to all inertial reference frames [21]. This theory is termed *special relativity*, since it dealt with inertial reference frames only.

However due to the constant velocity of light for all inertial observers, the transformation laws from one inertial frame to another must be different from that of the Galilean/Newton transformations. This is most evident when one considers the afore mentioned situation of two reference frames, one at rest and one moving with constant linear velocity which have a coinciding origin at a certain time t. Now supposing that as the origins coincide, a light source radiates a pulse, then at a later time the wave front will occupy a sphere of radius $c\Delta t$. Due to the constant velocity of light for both frames, the resulting equations of the two spheres as seen from each frame, must be

$$\Delta(x_1)^2 + \Delta(x_2)^2 + \Delta(x_3)^2 = c^2 \Delta t^2 \tag{4.11}$$

and

$$\Delta(x_1')^2 + \Delta(x_2')^2 + \Delta(x_3')^2 = c^2 \Delta t'^2 \tag{4.12}$$

The transformations from one frame to another will now indeed result in different *observed* length and time measurements in the direction of motion, but with each observers measurements being as valid as any other.

The special relativistic transformations are such that

$$x_1' = \gamma(x - vt) \tag{4.13}$$

$$x_2' = x_2 \tag{4.14}$$

$$x_3' = x_3 \tag{4.15}$$

$$t' = \gamma(t - \frac{vx}{c^2}) \tag{4.16}$$

Where the derived transformation factor is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4.17}$$

Which is identical to that derived by Lorentz and Fitzgerald. However, there is now a completely different viewpoint/understanding.

On inspection of the special relativistic transformations it is clear that space and time are inextricably linked. Also if $v > c$ then both x' and t' would be imaginary, the conclusion is that no observer can possess a velocity greater than that of light relative to any observer.

4.0.13 The Space-Time Continuum

The mathematician Hermann Minkowski realised that if the four quantities (three spacial and one temporal) were geometrically represented in a four dimensional Cartesian/Euclidean *space-time* [31] (a *space-time continuum*), such that

$$x_0 = ict \tag{4.18}$$

$$x_1 = x_1 \tag{4.19}$$

$$x_2 = x_2 \tag{4.20}$$

$$x_3 = x_3 \tag{4.21}$$

Where $i = \sqrt{-1}$

then one can recover the relativistic transformations (since the tangent of the angle of rotation $= \frac{iv}{c}$) by a simple rotation of the coordinate axes, which is of the form 3.2.

It is a matter of definition as to whether the factor i is chosen to be with the spatial coordinates or with the temporal coordinate.

The scalar invariant quantity (evident from 4.11 and 4.12 and using 4.18) in four dimensional space-time analogous to the interval between two points in ordinary space is thus

$$s^2 = \Delta(x_0)^2 + \Delta(x_1)^2 + \Delta(x_2)^2 + \Delta(x_3)^2 \tag{4.22}$$

which represents the interval between two *events* in space-time.

Thus, under a transformation analogous to 3.4, one has

$$s^2 = \sum_m \Delta(x_m)^2 = \sum_m \Delta(x'_m)^2 \tag{4.23}$$

Where m now ranges from 0 to 3.

Alternatively by use of the infinitesimal interval and metric tensor one can take the analogous form of 3.10 or 3.22 for flat space-time with Cartesian coordinates

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{4.24}$$

where now, with units of distance and time chosen such that the velocity of light is unity, then one can define (give a signature to) $g_{\mu\nu}$ such that

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.25}$$

where again the choice of sign to either the spacial or temporal is a matter of definition.

An equivalent method would also be to allow the 1's in the $g_{\mu\nu}$ to remain all positive and define contravariant and covariant components such that

$$X_0 = X^0 \qquad X_1 = -X^1 \qquad X_2 = -X^2 \qquad X_3 = -X^3 \tag{4.26}$$

The invariant scalar quantity (s) of 4.24, the space-time interval or *proper time interval*, may be used to construct a velocity vector v^μ with the correct space-time transformations. So the velocity vector is $v^\mu = \frac{dx^\mu}{ds}$.

A momentum vector p^μ may then also then be constructed with a mass m such that $p^\mu = mv^\mu$, the p^0 component of which is energy, therefore one has conservation of energy and momentum combined [1].

A material energy tensor or stress energy tensor $T^{\mu\nu}$ may also be constructed whereby one has the flux of the $\mu'th$ component of the momentum vector across a surface and as such in order for conservation of energy and momentum one must have $T^{\mu\nu}_{,\sigma} = 0$ [30].

Inspection of 4.24 shows that ds^2 can have three distinct outcomes.

1) If $ds^2 > 0$ (s is real) then the proper time interval between two events is the ordinary time interval measured in a frame in which the events occur at the same space point. The interval is said to be *timelike* and it is possible for a material body to be present at both events.

2) If $ds^2 < 0$ (s is imaginary) then it is not possible for a material for a material body to be present at both events. The interval is said to be *spacelike*.

3) If $ds^2 = 0$ ($s = 0$) Then only a light pulse can be present at both events. The interval is said to be null.

So one can see now, that by expressing all physical laws as *tensor equations* in a four dimensional Euclidean space-time then the covariance of these laws from one inertial frame to another can be guaranteed.

The inclusion of *accelerated* frames of reference into the theory of relativity seen in the following chapter will lead to a *general* theory of relativity and hence Einstein's law of gravity, where the space-time becomes *curved*.

Chapter 5

GENERAL RELATIVITY

5.0.14 The Equivalence Principle

The theory of Special Relativity is a theory based solely on the relative *uniform* motion of observers, where it was concluded that all inertial systems are equivalent for the description of physical phenomena.

If accelerated reference frames are to be considered then the phenomena of gravity (4.2) is naturally included.

Einstein realised that if one imagined an observer to be in, say, a rocket ship (driven by an engine), in deep outer space (far away from any massive bodies) then if the rocket ship has an acceleration ($\mathbf{a}$) then any unsupported particles inside the ship would have, as it would appear to the observer in the ship, an acceleration parallel to the acceleration of the ship. But knowing that the ship itself is accelerating (relative to any inertial frame) then the accelerated motion of the particles is simply attributed to this fact. However one may also treat the ship's observer to be in an inertial frame and as such the particle's motions are treated as being subjected to *inertial forces* (as mentioned in the previous chapter) acting parallel to the ship's motion. The inertial force on a particle of mass m would be $-m\mathbf{a}$

Similarly, if the ship is rotating about its center, the observer in the ship may assume an inertial frame and employ an inertial force (centrifugal) to act on the particles.

Now due to the fact that the inertial and gravitational mass are found to be equivalent, then the inertial forces can be thought of as arising from the presence of gravitational fields, or conversely a freely falling particle in a gravitational field can be considered to be at rest.

This is the *equivalence principle* [21] [30] which allows every observer to treat their reference frame as inertial and as such all observers become equivalent. Quite simply accelerated reference frames can now be treated as inertial frames and as such the Special Theory of Relativity has now become a *general* theory since *all* motion is now relative.

In the first example of the accelerated ship, where the observer takes himself to be at rest, then he will observe all particles/bodies (galaxies included) to have an acceleration $-\mathbf{a}$ to himself and he must attribute the cause of this acceleration to the presence of a uniform gravitational field affecting the bodies which extends over the whole of space.

Similarly, in the second example of the rotating ship, when the inertial frame is employed for the observer in the ship, it is the rotation of the bodies throughout the universe which are responsible for the gravitational field within the ship.

Quite simply the gravitational field is always present throughout all of space but by choosing a frame relative to which all the distant masses are at rest one can reduce the complication of the situation by this description. This is analogous to considering a distribution of electric charge, by choosing a frame relative to which the charge is at rest, then this omits the effect of the magnetic field and so makes calculations simpler.

This relates back to Mach's principle. However this is yet to be fully incorporated within the General Theory of Relativity (GTR) [30].

5.0.15 Curved Space-Time

Einstein also realised that, supposing one imagined a space station in the shape of a wheel, which again was in deep outer space (far away from any massive bodies) and the station was rotating with a constant angular velocity and attached to it there was an observer. Now suppose there was also an observer nearby who was not rotating, in other words this observer would be in an inertial frame relative to the rotating station. The observer attached to the station (who could also consider himself at rest) can, by use of a rigid measuring rod, measure out the radius (r) and circumference (c) of the circular station. He will obtain a result such that

$$\frac{c}{2r} = \pi \tag{5.1}$$

which is the standard result for Euclidean geometry.

The observer not attached to the station, watching the measurement being carried out, will agree with the measurement concerning the radius. However, due to the laws of special relativity, during the process of measurement of the circumference the unattached observer will record a length contraction of the measuring rod in the direction of motion parallel to his rest frame, as such he will disagree with the measurement of the circumference and he will obtain a result such that

$$\frac{c}{2r} < \pi \tag{5.2}$$

and therefore the laws of Euclidean geometry are in conflict with this.

When the Equivalence Principle is considered concerning this result, the conclusion is such that, relative to a frame at rest in a gravitational field, the geometry is not Euclidean. In the same way, from the laws of Special Relativity, there is the time dilation to consider.

The net result is that the space-time geometry of GTR is not Euclidean, whereby the gravitational field determines the geometry of the space-time continuum.

Now the situation as regards both the uniformly accelerating ship's gravitational field and the gravitational field of the rotating station is a little different from the gravitational field which surrounds a massive body. In the former two cases it is possible to transform *completely* away from the gravitational field (ie one can find an inertial reference frame relative to which the field vanishes and for which the space-time geometry is Euclidean). In the last case it is not possible to transform completely away in such a manner. However one *can* find a small region of space for a small duration of time (*local space-time*) for which the geometry will be *approximately* Euclidean, ie a small freely falling reference frame in a gravitational field which surrounds a massive object. Such an infinitesimal region of the space-time continuum of GTR would therefore approximate the space-time continuum of special relativity.

5.0.16 Replacing Newton's Law of Motion

Since the space-time continuum is now curved in GTR, all physical laws and equations must be expressed as *general* tensor quantities with covariant derivatives, as shown from 3.20 to 3.63, where the mathematical arguments are followed as in [17] with the extension from ordinary space geometry to space-time geometry. This curved space-time geometry is built in exactly the same way as the curved space geometry but the invariant quantity of the interval is now that of 4.24. So one has the invariant quantity between two events in space-time such that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{5.3}$$

obviously with the same $g_{\mu\nu}$ as 4.25.

So, with reference to the geodesic equation of 3.43, which was constructed from the change of a contravariant vector under parallel transport, equation 3.41, one can now construct an analogous geodesic for the space-time continuum. If one defines a velocity vector such that $v^\mu = \frac{dz^\mu}{ds}$ where s is the proper time, then one has the time-like $(v^\mu v_\mu > 0)$ geodesic equation

$$\frac{dv^\mu}{ds} + \Gamma^\mu_{\nu\sigma} v^\nu v^\sigma = 0 \tag{5.4}$$

or

$$\frac{d^2 z^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dz^\nu}{ds} \frac{dz^\sigma}{ds} = 0 \tag{5.5}$$

Now since the space-time is curved, Einstein assumed that a particle/body that is not acted upon by any forces, but is in a gravitational field, will follow the path of such a time-like geodesic. Essentially Equations 5.4 or 5.5 now replace Newton's first law of motion.

Following the earlier understanding of a null interval, the path of a pulse of light is now seen to be a null geodesic.

5.0.17 Einstein's Law of Gravity

Einstein also assumed that in empty space (no matter present and no other physical fields present except the gravitational field) then the gravitational field is given by

$$R_{\mu\nu} = 0 \tag{5.6}$$

which is the Ricci tensor 3.60 for space-time set equal to zero, and this is Einstein's law of gravitation. The use of the Ricci tensor for this law follows from the fact that, as mentioned earlier, it is not possible to transform completely away from a gravitational field surrounding a massive body and as such it is not possible to find a coordinate system where the $g_{\mu\nu}$ are constants. This would be the case if

$$R_{\mu\nu\sigma\rho} = 0 \tag{5.7}$$

with reference to 3.59 but now for space-time.

Now Equation 5.6 will still satisfy flat space with the condition that 5.7 holds.

So the Law of Gravitation is such that

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0 \tag{5.8}$$

and since the space-time version of 3.38 is

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}) \tag{5.9}$$

it is clear that Einstein's Law of Gravitation consists of a set of non-linear second order differential equations, when one considers the $g_{\mu\nu}$. If the $g_{\mu\nu}$ are looked on as potentials describing the gravitational field then there is a definite similarity between Einstein's theory and Newton's law.

5.0.18 The Newtonian Approximation

Newton's law of gravity explains the motions of most of the planets in the solar system very well and as such when one approximates GTR by introducing certain limits, then GTR reduces to Newton's law as would be expected.

The three conditions for the approximation are

1) The gravitational field is static.

2) All particles/bodies are moving slowly w.r.t the velocity of light.

3) the gravitational field is weak.

So, considering a static gravitational field, one has

$$g_{\mu\nu,0} = 0 \tag{5.10}$$

Also if the space curvature is a function of the space coordinates only, and similarly the time curvature is a function of the time coordinates only, and since

$$g_{\mu\nu} = y^n_{,\mu} y_{n,\nu} \tag{5.11}$$

then

$$g_{m0} = g_{0m} = 0 \tag{5.12}$$

where m (and also in the following n) take on the values $1, 2, 3$.

From 5.12 one also has $g^{m0} = g^{0m} = 0$ and $g^{00} = \frac{1}{g_{00}}$

This now means that from 5.10 and 5.12 one has

$$\Gamma_{m0n} = 0 \tag{5.13}$$

and therefore

$$\Gamma^{m}_{0n} = 0 \tag{5.14}$$

Now if one divides through equation 5.3 by ds^2 and one has the velocity vector $v^\mu = \frac{dx^\mu}{ds}$ then one has

$$1 = g_{\mu\nu} v^\mu v^\nu \tag{5.15}$$

With the use of 5.12, this then gives

$$1 = g_{00}(v^0)^2 + g_{mn} v^m v^n \tag{5.16}$$

but considering the approximation of slowly moving particles/bodies then $v^m << v^0$ and v^m is a small quantity of the first order and, as such, the quadratic quantities in the second term on the right hand side of 5.16 can be neglected. This leaves

$$g_{00}(v^0)^2 = 1 \tag{5.17}$$

Now since (as mentioned earlier) a particle/body is assumed to travel along the geodesic

$$\frac{dv^\mu}{ds} = -\Gamma^\mu_{\nu\sigma} v^\nu v^\sigma \tag{5.18}$$

which has now led to

$$\frac{dv^m}{ds} = -\Gamma^m_{\nu\sigma} v^\nu v^\sigma \tag{5.19}$$

using 5.14 this now becomes

$$\frac{dv^m}{ds} = -\Gamma^m_{00}(v^0)^2 - \Gamma^m_{nq} v^n v^q \tag{5.20}$$

Again the second term on the right can be neglected, so this leaves

$$\frac{dv^m}{ds} = -\Gamma^m_{00}(v^0)^2 \tag{5.21}$$

Multiplying through by $g_{mn}g^{mn}$ gives

$$\frac{dv^m}{ds} = -g^{mn}\Gamma_{n00}(v^0)^2 \tag{5.22}$$

Now with the use of 5.10 and 5.12 one has

$$\Gamma_{n00} = -\frac{1}{2}g_{00,n} \tag{5.23}$$

and substituting this into 5.22 gives

$$\frac{dv^m}{ds} = \frac{1}{2}g^{mn}g_{00,n}(v^0)^2 \tag{5.24}$$

Also, when small terms are neglected, one has

$$\frac{dv^m}{ds} = \frac{dv^m}{dx^\mu}\frac{dx^\mu}{ds} = \frac{dv^m}{dx^0}v^0 \tag{5.25}$$

So by equating 5.24 and 5.25 and using 5.16, one gets

$$\frac{dv^m}{dx^0} = g^{mn}(\sqrt{g_{00}})_{,n} \tag{5.26}$$

Multiplying through by g_{mn} leaves

$$\frac{dv_m}{dx^0} = (\sqrt{g_{00}})_{,n} \tag{5.27}$$

Now this shows that a particle/body moves as if it was under the influence of a potential of $\sqrt{g_{00}}$. One now needs to take Einstein's law of gravitation itself and introduce the limits to obtain a condition for the potential and therefore compare it to Newton's law. So, Einstein's law is such that

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0 \tag{5.28}$$

Now, with the approximation that the gravitational field is weak, there are small first order corrections to the $g_{\mu\nu}$, hence the $g_{\mu\nu,\sigma}$ are small and hence so are the $\Gamma^{\mu}_{\nu\sigma}$ and as such the quadratic terms in 5.28 can be neglected. This leaves

$$\Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} = 0 \tag{5.29}$$

which leads to

$$g^{\rho\sigma}(g_{\rho\sigma,\mu\nu} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\mu\nu,\rho\sigma}) = 0 \tag{5.30}$$

with the earlier conditions now giving that $\mu = \nu = 0$ (since one is dealing with the 00 components only, stemming from the geodesic acceleration, and as such one is finding R_{00}), and also since the $g_{\mu\nu}$ are independent of x^0 ($g_{\mu\nu,0} = 0$), one gets

$$g^{mn}g_{00,mn} = 0 \tag{5.31}$$

Now as stated earlier, all the equations of physics must be written as tensor equations. The d'Alembert equation for a scalar Φ, with use of 3.47 and 3.50 is therefore

$$g^{\mu\nu}\Phi_{:\mu:\nu} = g^{\mu\nu}(\Phi_{,\mu\nu} - \Gamma^{\alpha}_{\mu\nu}\Phi_{,\alpha}) = 0 \tag{5.32}$$

In the weak field approximation, the covariant derivative may become an ordinary derivative, so one has

$$g^{\mu\nu}\Phi_{,\mu\nu} = 0 \tag{5.33}$$

and using the earlier approximation of a static gravitational field, this reduces to the Laplace equation of

$$g^{mn}\Phi_{,mn} = 0 \tag{5.34}$$

Therefore, a comparison of 5.31 and 5.34 shows that here the Laplace equation is satisfied with g_{00} as the potential.

Now, as mentioned, the weak field approximation means there are small corrections to the g_{00}, therefore the g_{00} may be written

$$g_{00} = 1 + 2\Phi \tag{5.35}$$

where Φ is small.

therefore a binomial expansion of $\sqrt{g_{00}}$ to the first order gives

$$\sqrt{g_{00}} = 1 + \Phi \tag{5.36}$$

Equation 5.36 can now be substituted into 5.26, which yields

$$\frac{dv^m}{dx^0} = g^{mn}\Phi_{,n} \tag{5.37}$$

and since g^{mn} has diagonal elements which are approximately -1, this gives one

$$acceleration = -\nabla\Phi \tag{5.38}$$

where $\nabla = grad$ and Φ is identified with the Newtonian potential of 4.3.

This analysis shows that Newton's Law is recovered when Einstein's theory is considered in the weak and static field limit and the $g_{\mu\nu}$ are looked on as potentials.

Einstein's Theory of Gravitation (in 5.28) is expressed with no matter present and it has just been shown that the theory expressed in this manner will reduce to Newton's law when certain limits are considered. It can also be shown that Einstein's theory may be applied, in the special case where there is a static spherically symmetric field produced by a spherically symmetric body at rest, to very accurately describe the orbit of Mercury [17] [30], a motion that Newton's law could not describe.

5.0.19 Einstein's Law Including Matter

Einstein's law may be modified to include the presence of matter, as outlined in the following [17], so taking

$$R_{\mu\nu} = 0 \tag{5.39}$$

this means that

$$R = 0 \tag{5.40}$$

Now, when one considers the Bianci relations for the Ricci tensor (from 3.63) which are found to be

$$(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{:\sigma} = 0 \tag{5.41}$$

and since from 5.39 and 5.40 it is clear that

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 0 \tag{5.42}$$

then it is best to work with 5.42 in order to obtain conservation laws when considering the inclusion of matter.

If the tensor $Y^{\mu\nu}$ is introduced, which is to be associated with the presence of matter (and with a coefficient of -8π, as will later become apparent in order to give this tensor the properties of the density and flux of energy and momentum), then one has

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi Y^{\mu\nu} \tag{5.43}$$

and therefore from 5.41 this must mean that

$$Y^{\mu\nu}_{:\sigma} = 0 \tag{5.44}$$

It should be noted here that only in flat space-time would there be the usual expected conservation of matter, i.e. $Y^{\mu\nu}_{,\sigma} = 0$. In the curved space-time the covariant derivative means there is an additional term involving $\Gamma^{\mu}_{\nu\sigma}$. Due to this additional term the conservation law can only be reconciled if the gravitational field itself is assumed to possess energy and momentum.

Now, to continue, if one considers a distribution of matter with velocities varying continuously throughout, then one has the velocity vector

$$v^{\mu} = \frac{dz^{\mu}}{ds} \tag{5.45}$$

where dz^{μ} are the coordinates of a molecule of matter.

If then a scalar field ρ is introduced such that the vector field ρv^μ is the density and flux of the matter, then with the energy density, energy flux, momentum density and momentum flux considered, one has the density and flux of energy and momentum given by

$$\rho v^\mu v^\nu = T^{\mu\nu} \tag{5.46}$$

where $T^{\mu\nu}$ is the material energy tensor or stress energy tensor.

Now it can be shown that [17],

$$T^{\mu\nu}_{:\sigma} = 0 \tag{5.47}$$

So by substitution of 5.47 into 5.43 one has

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = k\rho v^\mu v^\nu \tag{5.48}$$

where k is a constant.

By contraction and limits introduced as in the earlier Newtonian approximation (as outlined below), then 5.48 becomes

$$\nabla^2 \Phi = -\frac{1}{2} k\rho \tag{5.49}$$

Now in order for 5.49 to be compared with Poisson's equation 4.4, then $k = -8\pi$.

This analysis shows that Einstein's equation of gravitation, in the presence of matter, becomes

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi\rho v^{\mu}v^{\nu} \tag{5.50}$$

There are obviously two other forms of 5.50 obtained by lowering of indices. If one substitutes for 5.46 and lowers indices, this yields

$$R^{\mu}_{\nu} - \frac{1}{2}g^{\mu}_{\nu}R = -8\pi T^{\mu}_{\nu} \tag{5.51}$$

and then again

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu} \tag{5.52}$$

Now, since $g^{\mu}_{\mu} = 4$, contraction of 5.51 yields

$$R = 8\pi T \tag{5.53}$$

So substitution of 5.53 into 5.52 and rearrangement gives

$$R_{\mu\nu} = 8\pi(\frac{1}{2}g_{\mu\nu}T - T_{\mu\nu}) \tag{5.54}$$

Now, in the static, weak field and slowly moving approximations, since one is considering simple, stable matter/energy distributions that do not carry a net flow of force, momentum or energy through space, then there is only one component (energy density) of the stress energy tensor which is not zero and so one has

$$T_{00} = \rho_0 v_0 v_0 = T \tag{5.55}$$

For the weak field, one will also have $g_{00} \approx 1$ and so 5.54 will reduce to

$$R_{00} = -4\pi T \tag{5.56}$$

and as in the earlier calculation for R_{00} there is a factor g^{mn} which is equal to minus unity, and therefore 5.56 will reduce to Poisson's equation as expected.

The law of gravitation with the inclusion of matter will be the one, when modified in Chapter Seven, which will be used in the analogous approximation analysis. A point source solution will then be calculated.

Einstein's Law of Gravitation has had tremendous success in describing a wide range of local physical phenomena, as mentioned in Chapter Two. However unless a hypothesised new form of matter can be detected and therefore proved to exist, GTR may need to be modified in order to explain the physical observed motions of galaxies, as will be seen in the next chapter.

Chapter 6

THE MOTIONS OF GALAXIES

6.0.20 Rotational Velocities

As seen in the previous chapter Einstein's General Theory of Relativity reduces to Newton's law of gravity when certain approximations are introduced into the field equations.

The fact that this Newtonian approximation can be achieved is to be expected since Newton's law of gravitation explains the motions of the planets in the solar system very well except for Mercury.

Of all the planets, Mercury has the closest orbit around the sun, where obviously the gravitational field is the strongest. The rest of the planets have orbits where the gravitational field is relatively weak and this is then consistent with the Newtonian approximation of Einstein's law in the weak field limit.

So, when considering *weak* accelerations it is natural to assume that any gravitational system should behave in a manner which follows Newton's law of gravity, and, as such, the variation in orbital circular (rotational) velocities reflecting the mass distribution of such a system should, away from the center, follow the standard Keplerian decline given by

$$V = \sqrt{\frac{Gm}{r}} \tag{6.1}$$

where V is the rotational velocity, r is the distance from the center G is the gravitational constant and m is the mass.

This Keplerian decline shows the decrease in rotational velocities as a function of distance from the center which the planets in the solar system follow the further away they are from the sun.

Now, during the 1930's, not long after galaxies were actually recognised to be extra-galactic in nature, Fritz Zwicky measured the velocity dispersion of galaxies in clusters of galaxies, where the accelerations are weak, and found it was far to high for these systems to remain stable for a substantial length of time [58]. Quite simply, according to Newton's law, these systems should be flying apart.

Some forty years later the rotational velocities of stars in individual spiral galaxies were measured [44]. These stars are mainly on approximately circular orbits around the center of the galaxy and again, since far from the center, the strength of accelerations is weak, then these systems should also show a Keplerian decline for the rotational velocities (of the stars) in the same manner as for the planets in the solar system.

However when the rotational velocities of these stars were calculated, by taking their measured Doppler shifts and comparing these quantities with their respective distance from the galactic center, and the total mass of such systems being that of the observed stars and gas, they were again found to be too high for these systems to remain stable. This phenomena was seen to happen over and over again as each galaxy was studied [44]. The individual galaxies themselves should, according to Newton's law, be flying apart.

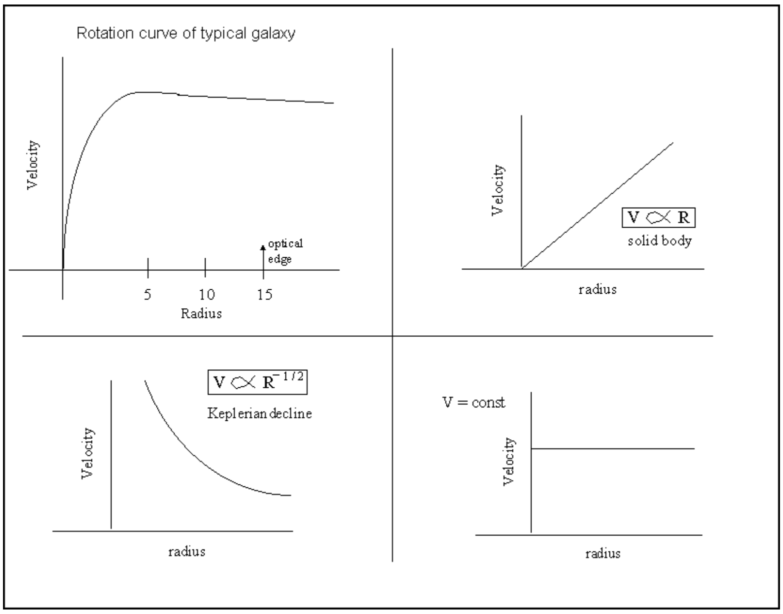

FIGURE 6.1: Rotation Curves

Obviously the rotational velocities of the stars in a typical galaxy may be represented graphically as can be seen in fig. 6.1 , where the Keplerian decline, solid body velocity and constant velocity are also graphed.

So, as can be seen in fig. 6.2 rather than follow the expected Keplerian decline, in a typical galaxy the rotation curve tends to remain approximately flat with

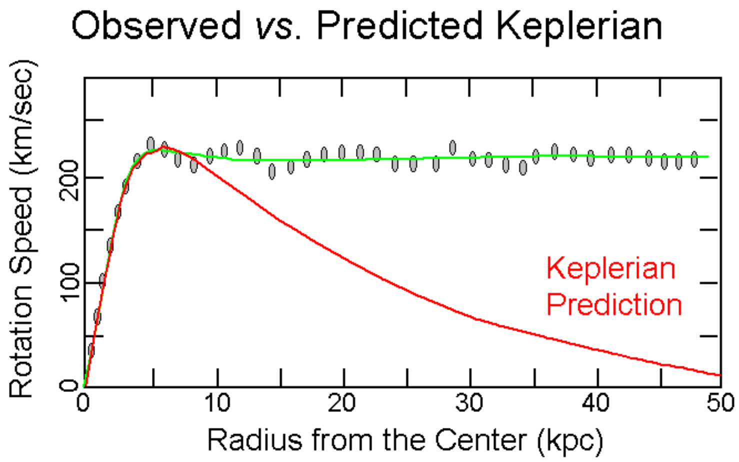

FIGURE 6.2: Typical Rotation Curve

increasing distance from the center (the velocity becomes approximately constant) and one has a 'flat rotation curve'.

So the question is now, how does one explain these observed motions?

The two most obvious possible solutions to the problem of explaining the flat rotation curves are

1) The inclusion of extra mass into the calculations when using the dynamical laws.

2) A modification of the dynamical laws themselves.

6.0.21 Dark Matter

The first possible solution was initially suggested by Zwicky and subsequently followed up by Peebles [42].

The Missing Mass Hypothesis (MMH) [45] postulates that some of the missing mass (Dark Matter) may be in the form of ordinary baryonic matter that has so far been unobserved but if the MMH is to fully explain the rotational dynamics of galaxies then the vast majority of the missing mass must be in the form of as yet unknown and undetected form of matter, one that does not emit and absorb radiation as ordinary baryonic matter does. This latter new form of matter was thought to have two possible forms, cold and hot.

Hot Dark Matter (HDM) is thought to be composed of particles that have zero or nearly zero mass (the main contender being the neutrino). Since the Special Theory of Relativity requires that massless particles move at the speed of light and nearly massless particles move at nearly the speed of light, this means that such very low mass particles must move at very high velocities, and therefore (by the kinetic theory of gases) form very hot gases.

Now, on the other hand, Cold Dark Matter (CDM), is thought to be composed of particles which are sufficiently massive enough causing them to only move at sub-relativistic velocities, and therefore they form much colder gases.

The significance of the difference between Hot and Cold Dark Matter is important in the DM model in structure formation explaining the Cosmic Background Radiation (CBR), mentioned later.

In the following the reference to Dark Matter (DM) will be that of the new non-baryonic matter, i.e the CDM.

The new hypothesised particles of DM are mainly thought to be in the form of Weakly Interacting Massive Particles (WIMPS) [54] [13] [51], since these particles are thought to hardly ever interact with ordinary baryonic matter.

So ever since the idea of DM was postulated the hunt has been on to discover this elusive substance [52]. However, to date, all experiments designed to detect DM have proved fruitless. The latest experiment (designed at a cost of over a billion pounds) to try and detect DM is the LUX experiment in Sanford, South Dakota. The experiment is being performed at a location about a mile underground in order to reduce interference from any weakly interacting baryonic particles such as cosmic rays.

The general premise and principle of the experiment is that, if WIMPS exist, then as they pass through a large volume of liquid Xenon, a collision with an Xenon atom is expected, the correct predicted energy detection of such a collision would therefore possibly prove its existence.

It is presumed that the earth experiences a Dark Matter *'wind'* (a flux of Dark Matter) passing through it due to the earth's motion through the postulating Dark Matter halo of our galaxy. The model calculates that there are nearly one billion WIMPS per square meter per second passing through the earth and it is expected that these WIMPS have a small yet measurable interaction cross section with ordinary nuclei (baryonic matter). In other words there is a small but finite probability of an incoming WIMP scattering off a laboratory target in such a way that it can be detected. The scattering event would be characterised by a recoil energy of a few to tens of KeV, which is a very small but theoretically observable signal. Such a signal takes one of three forms: Scintillation light, ionization of an atom within the target or heat energy in the form of phonons.

Xenon is a natural choice as a medium for these direct detection experiments because it is easy to read out signals from two of these channels. Energy deposited in the scintillation channel should be easily detectable since Xenon is transparent to its own characteristic 175 mm scintillation. The assumed energy deposited in the ionization channel should likewise be easily detectable, since ionization electrons under the influence of an applied electrical field can drift through Xenon for relatively large distances - up to several meters.

Furthermore, the ratio of the energy which would be deposited in these two channels would be a powerful tool for discriminating between nuclear recoils such as WIMPS and neutrons (which are the signals of interest), and electric recoils such as gamma rays, which are the major source of background interference.

Xenon is also particularly good for the experiment due to its 'self shielding' properties. Liquid Xenon is very dense and therefore gamma and neutrons tend to attenuate within just a few centimetres of entering the target. As such, any particle that does not have enough energy to reach the sensor of the target has a high probability of undergoing multiple scatters, which would be easy to pick out and reject. This makes Xenon ideal for Dark Matter searches.

The first run of the experiment was carried out in 2013 for a duration of ninety days but absolutely no hint of DM was detected, and this trial run also ruled out any possible hints of DM from previous experiments [2]. The experiment is running again now in 2014, this time for a period of two hundred days.

As for the theoretical DM model itself [22], the distribution of the dark matter is thought to be spread out throughout the galaxy but mainly concentrated in a large halo surrounding the galaxy and as such the term 'Dark Matter Halo' includes the dark matter within the galaxy.

In order for the DM model to fit the observed rotational velocities, it is necessary for at least three free parameters, a minimum of two to describe the DM halo plus the stellar mass to light ratio.

To give a brief overview of the basic theoretical DM model it is perhaps first best to return to the afore mentioned observations by Zwicky. The original observations of Zwicky were of the cluster of galaxies named the 'Coma', where Zwicky estimated the potential energy stored in the cluster to be of the form

$$E = -\frac{3Gm^2}{5r} = -m\sigma^2$$

Where G is the gravitational constant, r denotes the radius of the cluster, m is the total mass of the cluster and σ denotes the three dimensional velocity dispersion, i.e.the root mean square speed(about the mean)of the cluster galaxies. The first equality is arrived at by approximating the galaxy cluster as a homogeneous sphere with constant mass density and the second equality is given by the Virial theorem - which for a stable, self gravitating, spherical distribution of equal mass objects (stars, galaxies),states that the total Kinetic Energy (KE), of the objects is equal to minus one half multiplied by the total gravitational Potential Energy (PE) i.e $KE = -\frac{1}{2}PE$.

From images of galaxy clusters Zwicky was able to estimate the radius, r, and the observable luminous mass, m_0, of the individual galaxies, spectroscopic observations were able to provide the galactic line of sight velocities and thus the velocity dispersion. The observed value for the velocity dispersion exceeded the predicted value (from the Newtonian dynamics) by a factor of around twenty. Now since $m \propto \sigma^2$ this indicated a *'dynamical'*, gravitating mass of the system $m_d \approx 400m_0$.

It is this obvious discrepancy that led Zwicky to assume that most of the mass within the galaxy clusters was not luminous and this resulted in his coining of the phrase 'Dark Matter'. Later observations revealed that the discrepancy was not as large as Zwicky had initially estimated. A large amount of the luminous matter within the galaxy cluster was in the form of Hot Gases which were only observable at X-Ray wavelength. However, even when this is taken in to account, there is still (for an average galaxy cluster), a substantial mass discrepancy of $m_d \approx 8m_0$

As mentioned earlier, some forty years after Zwicky's observations of galaxy clusters, it became clear that individual galaxies themselves showed a discrepancy between luminous and dynamical mass. As seen in equa 6.1 the expected rotational velocity (as a function of radius r from the galactic centre)for a galaxy is that of

$$V^2(r) = \frac{r\partial\phi}{\partial r} = \frac{Gm(r)}{r}$$

Where ϕ denotes the gravitational potential and $m(r)$ is the mass enclosed within r, which can be derived from integration over the mass density as a function of r, $\rho(r)$

Now since it is observed that the velocity rotation curves are flat i.e. $V \approx constant$, the DM theory postulates that the galaxies behave as if they are surrounded by halos of matter extending well beyond the visible components.Therefore, by inspection of 6.1 the indication is that of halos with density profiles $\rho(r) \propto \frac{1}{r^2}$ and as such a simple parameterisation of a DM halo is achieved by a mass density profile $\rho(r) = \rho_0 \frac{r_0}{r}^2$, where ρ_0 is a scaling factor of the dimension of a mass density and r_0 denotes a characteristic radius. This functional form of the density profile follows naturally from the assumption that a DM halo is a self gravitating isothermal ensemble of particles in equilibrium. As such it follows that *any* DM halo model for a given galaxy requires *at least* three free parameters i.e. the scaling parameters ρ_0 and r_0 together with the galaxy's mass to light ratio which is needed to estimate the luminous mass from its brightness. An example of a modern sophisticated DM model would be that of the Navarro-Frank-White profile.

The necessity of at least three free parameters for a DM model is in sharp contrast to that of Modified Newtonian Dynamics which has only one parameter - a constant of nature.

6.0.22 MOdified Newtonian Dynamics (MOND)

A modification regarding the second possible solution was put forward by Mordehai Milgrom in the early 1980's [37].

Milgrom considered a possible change in the proportionality of force and acceleration, since typical accelerations in galactic systems are many orders of magnitude smaller than those encountered in the solar system.

Milgrom introduced a constant a_0 which has the dimensions of acceleration (SI units). This constant was determined in a few independent ways [5] and found to have a value of order 10^{-10}. Now if accelerations (a) are much larger than a_0 ($a \gg a_0$) then the assumption is that standard Newtonian dynamics is a good approximation for the system and hence

$$a = \frac{mG}{r^2} \tag{6.2}$$

In the opposite limit $a \ll a_0$ then the acceleration is given by

$$\frac{a^2}{a_0} = \frac{mG}{r^2} \tag{6.3}$$

Now these two expressions may be interpolated to give

$$\mu(\frac{a}{a_0})a = \frac{mG}{r^2} = a_N \tag{6.4}$$

where a_N is the Newtonian acceleration, and the interpolating function $\mu(x)$ satisfies $\mu(x) \approx 1$ when $x \gg 1$ and $\mu(x) \approx x$ when $x \ll 1$.

The MOND phenomenology is thus much simpler than the DM model and cannot be adjusted to fit rotation curves in a way the DM model can. This is evident when one considers that a_0 is a constant of nature, yet *any* DM model necessarily requires (as seen earlier), at least *three* free parameters.

The modified Poisson equation for the MOND phenomenology [6] is

$$\bigtriangledown \cdot [\mu(\frac{\mid \bigtriangledown \phi \mid}{a_0}) \bigtriangledown \phi] = 4\pi\rho \tag{6.5}$$

Note for consistency with the tensor analysis the units have been chosen such that $G = 1$.

This modification was originally derived from the action principle for GTR [57]. In order for it to be fully consistent with 6.4, it is assumed there is no curl field present.

MOND is found to describe the motions of galaxies of all types of galaxies extremely well [16] [35] [23] [39] including spiral, elliptical and dwarf and it is also consistent with the Tully-Fisher relationship [53]. It therefore would appear to have a very strong case for being the correct model for the description of the motions of galaxies.

There are several other galactic and cosmological problems which have initially incurred the MMH and subsequently the DM model. These include the measurements on The Bullet Cluster [11], Cosmic Background Radiation (CBR) [50] and galactic lensing [14]. Although one's take on the evidence is somewhat subjective, pointing one way or another regarding which is a better explanation for these anomalies, DM or MOND [36], MOND is an equally good candidate [4] [34] [40].

6.0.23 Relativistic MOND

The main problem for MOND is it needs a relativistic extension. If a modification of GTR can be found which naturally leads to the MOND phenomenology then this would surely suggest the validity of MOND. There have been several attempts to modify GTR in this way including TeVeS [8] [49], Conformal Gravity [33] [32], and DM as a curvature effect [12], but unfortunately none have been wholly satisfactory [47] [48].

The following chapter describes a new attempt to modify GTR to this end. This modification of GTR is based on a novel simple physical assumption - that the space-time continuum is not only curved due to the presence of mass but there is also the effect of a local expansion of the space-time continuum.

Chapter 7

THE LOCAL EXPANSION OF SPACE-TIME

7.0.24 A Non-Riemannian Geometry

The previous chapters (excluding Chapter Six) have built up to and culminated in Einstein's GTR. The success of the theory at solar system scale has given the theory huge credibility. However the apparent failure of Newton's law to explain the large scale motions of galaxies suggests that possibly GTR is not complete and as such requires a modification which would lead to the correct predictions for the dynamics of galaxies.

The following work concerns the novel concept of *local space-time expansion* included within standard GTR. The concept is such that, not only is the geometry of space-time curved as in standard GTR (a Riemannian geometry for space-time), but there is *also* a local expansion of the space-time (a non-Riemannian geometry for the space-time). The non-Reimannian geometry which results is very similar to that of Weyl's. However Weyl attempted to associate the resulting extra mathematical term with the physical reality of the electromagnetic field. Einstein had objected to Weyl's theory claiming that there would now be an extra time factor to consider (the proper time would no longer equate to the atomic time). However it is interesting to note that this was a mainly philosophical argument since the small ambiguities of length comparisons were too small to be detected [19].

Weyl eventually developed two forms of his theory. Physical quantities may be represented in the form of a graph and this representation is most often used when one wishes to set out a mass of information in such a way that the eye can take it in in at a glance. This representation is of scientific ingenuity and obviously extremely helpful but it is not a test of the *'nature'* of the world but of the ingenuity of the mathematician. Weyl's second form of his theory is of this nature.

However, this is not the only use of a graphical representation. The representation of a *'conceptual'* mathematical space of any number of dimensions, and if desired a non-Euclidean geometry can be achieved in this manner and in this sense physics geometrised, this is essentially Einstein's view in his General Theory of Relativity as a *'natural'* (actual) geometry by using the space-time adaptation of Riemannian geometry.

Weyl's concern was such that electromagnetic phenomena itself (the quantity), was not represented with Einstein's theory. One possible way to incorporate electromagnetic quantities in to the theory was to assume that the Riemannian geometry assigned to actual space was not exact and that the true geometry was of a broader sense leaving room for the electromagnetic phenomena to be included in to the natural geometry of space-time (i.e the formulation of a non-Riemannian geometry).

To distinguish fully between the two forms of Weyl's theory, one needs to distinguish between a *'natural'* geometry and a *'world'* geometry. The natural geometry is the single *'true'* (actual) geometry in the sense understood by the physicist. The world geometry is the pure geometry applicable to a conceptual graphical representation of all the quantities concerned in physics. Thus in Weyl's first form of his theory, it is Einstein's natural Riemannian geometry which is amended to a broader natural non-Riemannian geometry.

In Chapter's three and five it was seen that in Riemannian geometry and the extended space time geometry, that as a vector undergoes a parallel displacement the length remains constant, there is no change in length. However as one will see in this chapter, one can construct a geometry in which a change of length occurs without leading one in to any type of contradiction. This results in a non-Riemannian geometry with an extra term appearing as compared with Einstein's analysis.

Weyl's non-Riemannian geometry is that of a geometry where a change of length occurs and it is mathematically very similar to the non-Riemannian geometry presented in this chapter. However, the physical interpretation is completely different.

Within this non-Riemannian geometry, is it possible to compare lengths (beside zero length) at different places? Since the result of the comparison will depend on the route taken in bringing the two lengths closer together. In Riemannian geometry the comparing of two lengths is taken for granted since the interval at any point has been assigned a definite value and this implies comparison with a standard.

The new geometry needs to be set up in such a way that comparisons can be made. This can be achieved by supposing that a definite but arbitrary *'gauge system'* has been adopted - in other words, at every point of space time a standard of interval length has been set up and every interval is expressed in terms of the standard *at the point where it is.* This avoids the ambiguity involved in transferring intervals from one point to another to compare with a single standard. Thus one is now dealing with transformation of a gauge system (gauge invariance), as well as transformation of co-ordinates.

Change of gauge is a generalisation of change of unit in physical equations, the unit being no longer a constant but an arbitrary function of position.

The only unit which needs to be considered is that of the interval. Co-ordinates are merely identification numbers and have no reference to the interval unit (note - if one changes the unit mesh of a rectangular co-ordinate system from one mile to one kilometre, one makes a change of co-ordinates not a change of gauge).

So for Weyl this now paved the way for a change in length to manifest itself physically as an alteration of the electro magnetic field i.e. to describe the phenomena of electromagnetism.

Weyl had proposed his theory shortly after Einstein had published his General Theory of Relativity and at that time it was thought that *all* the phenomena of mechanics had been traced to the $g_{\mu\nu}$ in Einstein's theory and thus the change in length according to Weyl must naturally be linked directly to what was left, namely the domain of electromagnetism.

Weyl's theory, involving a natural geometry. has died from an inanation rather than by direct disproof, in other words it had seemed an unnecessary speculation to introduce small ambiguities of length comparisons which are too small to be practically detected, merely to afford the satisfaction of geometrising the electromagnetic phenomena.

This leads neatly on to the concept of a local expansion of space time. Since it is now well established that the motions of galaxies do not fit with GTR/Newton, then perhaps the change in length does indeed have a mechanical effect. One assumes that the change in length is linked directly to a local physical expansion of space time itself and that this expansion can only be 'picked up' (detected) at large distances where it manifests itself in the observed mechanics of large systems such as galaxies. As such in the following theory the General Theory of Relativity remains unaltered for small systems e.g solar.

Perhaps the local expansion of space time is correlated to Quantum Mechanical effects on space time. For example, in simple terms, the indeterminate position of a particle may have the effect of local space time expansion and vice versa.

In the previous chapters the GTR has been derived whereby a flat higher dimensional space has been assumed (and then discarded) in order to find equations for the change (in *direction*) of a vector under parallel displacement within a curved space-time (following the arguments in [17]).

By introducing an expansion factor into the expression for the interval (distance measure) in the higher dimensional space one can formulate a situation whereby there is now found to *also* be a change in *length* of the vector under parallel displacement. This procedure is equivalent to introducing conformal geometry into the standard GTR analysis. Conformal geometry is concerned with the preservation of angles under transformation but *not* lengths. This change in length is to be related to the physical concept of the *local expansion of space-time*, thereby both the mathematics/geometry and physics are completely generalised.

The inclusion of the expansion factor results in a term additional to the standard Christoffel symbols. This extra term is present and consistent throughout the analysis and in the analogous Newtonian approximation it can be used to fit the observations of the dynamics of the galaxies, that is, the MOND phenomenology.

It can also be shown that the expansion term is negligible for small systems,-therefore standard GTR is not altered for the solar system. However the expansion term becomes the dominant term for large systems and therefore gives a physical basis for the observed MOND dynamics of galaxies. In fact also, throughout the forthcoming analysis when the expansion factor is set to unity the standard expressions for the GTR are recovered.

7.0.25 The Distortion Of Space-Time

The following treatment, not previously published to the best of the authors knowledge, is the same previous derivation for standard GTR but now with an expansion factor included.

So the interval in the higher dimensional space is initially

$$ds'^2 = h_{nm} dz'^n dz'^m \tag{7.1}$$

now the expansion is introduced (which is isotropic at a point z^m) such that $dz'^m = \sqrt{\alpha} dz^m$, the expansion factor is thus $\frac{1}{\sqrt{\alpha}}$ and is a function of position. The parameter α is thus linked directly to the local expansion of space-time.

This means 7.1 now becomes

$$ds'^2 = \alpha h_{nm} dz^n dz^m \tag{7.2}$$

It should be noted that the formulation here has no direct relation to Tetrads. A Tetrad is a set of axis introduced in to the General Theory of Relativity as a frame from which one can transform to the co-ordinate frame and vis versa in order for certain areas of physics, (e.g. Quantum Mechanics), to become more transparent.

In a Tetrad formulation the interval has the form

$$ds^2 = \gamma_{mn} e^m_\mu dx^\mu e^n_\nu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu$$

Where e^m_μ is defined to be the transformation co-efficients between the Tetrad frame and the co-ordinate frame.

Now the curvature is introduced in exactly the same way as before with a point $y^n(x)$ in the higher dimensional space corresponding to a point x^μ in the embedded curved four dimensional space-time.

So for two neighbouring points on the surface of the four dimensional space-time one has

$$\delta y^n = y^n_{,\mu}\delta x^\mu \tag{7.3}$$

and thus from 7.1 the squared distance between these two points is

$$\delta s'^2 = \alpha h_{nm}\delta y^n \delta y^m \tag{7.4}$$

From 7.3 this becomes

$$\delta s'^2 = \alpha h_{nm} y^n_{,\mu} y^m_{,\nu}\delta x^\mu \delta x^\nu \tag{7.5}$$

$$= \alpha y^n_{,\mu} y_{n,\nu}\delta x^\mu \delta x^\nu \tag{7.6}$$

Now since one also has

$$\delta s'^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu \tag{7.7}$$

the metric tensor is now given by

$$g_{\mu\nu} = \alpha y^n_{,\mu} y_{n,\nu} \tag{7.8}$$

where both curvature *and* expansion determine its components. If $\alpha = 1$ then the metric tensor is that of standard GTR.

In order for the interval to be invariant one needs

$$ds^2 = y^n_{,\mu} y_{n,\nu} dx^\mu dx^\nu \tag{7.9}$$

and as such in a weak gravitational field then $ds = dx^0$ that is the proper time becomes the atomic time, to ensure that this is still the case with the expansion factor included one can have both a gauge rescaling and a counterbalancing length rescaling present. Therefore equating the infinitesimal of 7.5 with 7.9 one has

$$\alpha ds^2 = ds'^2 = \alpha y^n_{,\mu} y_{n,\nu} dx^\mu dx^\nu \tag{7.10}$$

The formulation here (7.9 and 7.10) allows, for weak curvature, the proper time to become the atomic time, although the theory presented here does not necessarily need this constraint.

7.0.26 Parallel Displacement

Now the δx^μ of 7.3 is equivalent to a vector(components) A^μ in the physical space-time (the A^n being the corresponding vector in the higher dimensional space), thus

$$A^n = y^n_{,\mu} A^\mu \tag{7.11}$$

this vector is now shifted by parallel displacement to a neighbouring point $x + dx$, the vector does not now lie in the physical space-time, however (as discussed in Chapter Three) it can be projected back onto the surface by splitting the vector into a tangential part and a normal part and then discarding the normal part. So for the displaced vector one has

$$A^n = A^n_{tan} + A^n_{norm} \tag{7.12}$$

Now let K^μ denote the components of A^n_{tan} in the curved space-time, so from 7.11 one has

$$A^n_{tan} = K^\mu y^n_{,\mu}(x + dx) \tag{7.13}$$

Now since A^n_{norm} is defined to be normal to *every* tangential vector at $x + dx$ one has the scalar product

$$A^n_{norm} y_{n,\mu}(x + dx) = 0 \tag{7.14}$$

Therefore, by using 7.13 and multiplying 7.12 through by $y_{n,\nu}(x+dx)$ the normal term is discarded, leaving

$$A^n y_{n,\nu}(x+dx) = K^\mu y^n_{,\mu}(x+dx) y_{n,\nu}(x+dx) \tag{7.15}$$

substitution of 7.8 then gives

$$A^n y_{n,\nu}(x+dx) = \frac{1}{\alpha(x+dx)} g_{\mu\nu}(x+dx) K^\mu(x+dx) \tag{7.16}$$

$$= \frac{1}{\alpha(x+dx)} K_\nu(x+dx) \tag{7.17}$$

rearranging gives

$$K_\nu(x+dx) = A^n y_{n,\nu}(x+dx)\alpha(x+dx) \tag{7.18}$$

Taylor expansion of the RHS gives

$$K_\nu(x+dx) = A^n[(y_{n,\nu}(x) + y_{n,\nu\sigma}dx^\sigma)(\alpha(x) + \alpha_{,\sigma}dx^\sigma)] \tag{7.19}$$

with use of 7.11 and 7.8 this then becomes

$$K_\nu(x+dx) = A_\nu + [A_\nu(ln\alpha)_{,\sigma} + A^\mu \alpha y^n_{,\mu} y_{n,\nu\sigma}]dx^\sigma \tag{7.20}$$

Since the K_ν is the result one gets when the vector A_ν is parallel displaced to the point $x + dx$ one can write

$$K_\nu - A_\nu = dA_\nu \tag{7.21}$$

where dA_ν is the change in the vector A_ν under parallel displacement, as such

$$dA_\nu = [A_\nu(ln\alpha)_{,\sigma} + A^\mu \alpha y^n_{,\mu} y_{n,\nu\sigma}]dx^\sigma \tag{7.22}$$

or with the use of $g_{\mu\nu}A^\mu = A_\nu$ one has

$$dA_\nu = A^\mu[g_{\mu\nu}(ln\alpha)_{,\sigma} + \alpha y^n_{,\mu} y_{n,\nu\sigma}]dx^\sigma \tag{7.23}$$

this can be compared with the same manner of change (recall 3.36) in the standard GTR which is

$$dA_\nu = A^\mu y^n_{,\mu} y_{n,\nu\sigma} dx^\sigma \tag{7.24}$$

Now the y^n refers to the higher dimensional space, and this reference can be made to disappear by differentiation, manipulation and subsequent substitution of 7.8 as follows

$$g_{\mu\nu,\sigma} = \alpha[y^n_{,\mu} y_{n,\nu\sigma} + y_{n,\nu} y^n_{,\mu\sigma}] + y^n_{,\mu} y_{n,\nu} \alpha_{,\sigma} \tag{7.25}$$

$$= \alpha[y^n_{,\mu} y_{n,\nu\sigma} + y^n_{,\nu} y_{n,\mu\sigma}] + y^n_{,\mu} y_{n,\nu} \alpha_{,\sigma} \tag{7.26}$$

Interchanging μ and σ one gets

$$g_{\sigma\nu,\mu} = \alpha[y^n_{,\sigma} y_{n,\nu\mu} + y^n_{,\nu} y_{n,\sigma\mu}] + y^n_{,\sigma} y_{n,\nu} \alpha_{,\mu} \tag{7.27}$$

Interchanging ν and σ (from 7.25) one gets

$$g_{\mu\sigma,\nu} = \alpha[y^n_{,\mu} y_{n,\sigma\nu} + y^n_{,\sigma} y_{n,\mu\nu}] + y^n_{,\mu} y_{n,\sigma} \alpha_{,\nu} \tag{7.28}$$

Now from 7.8 one has for the last terms in 7.25, 7.27 and 7.28

$$y^n_{,\mu} y_{n,\nu} \alpha_{,\sigma} = \frac{g_{\mu\nu}}{\alpha} \alpha_{,\sigma} = g_{\mu\nu} (ln\alpha)_{,\sigma} \tag{7.29}$$

Interchange of μ and σ gives

$$y^n_{,\sigma} y_{n,\nu} \alpha_{,\mu} = \frac{g_{\sigma\nu}}{\alpha} \alpha_{,\mu} = g_{\sigma\nu} (ln\alpha)_{,\mu} \tag{7.30}$$

Interchange of ν and σ gives

$$y^n_{,\mu}y_{n,\sigma}\alpha_{,\nu} = \frac{g_{\mu\sigma}}{\alpha}\alpha_{,\nu} = g_{\mu\sigma}(ln\alpha)_{,\nu} \tag{7.31}$$

So if one takes 7.25+7.28−7.27 and divides by two, one has the standard Chrstoffel symbols (recall 3.38). These are now seen to be of the form

$$\Gamma_{\mu\nu\sigma} = \alpha y^n_{,\mu}y_{n,\nu\sigma} + \frac{1}{2}[g_{\mu\nu}(ln\alpha)_{,\sigma} + g_{\mu\sigma}(ln\alpha)_{,\nu} - g_{\sigma\nu}(ln\alpha)_{,\mu}] \tag{7.32}$$

7.0.27 The Expansion Symbols

The *Expansion* symbol E is now introduced such that

$$E_{\mu\nu\sigma} = \frac{1}{2}[g_{\mu\nu}(ln\alpha)_{,\sigma} + g_{\mu\sigma}(ln\alpha)_{,\nu} - g_{\sigma\nu}(ln\alpha)_{,\mu}] \tag{7.33}$$

so 7.32 is now

$$\Gamma_{\mu\nu\sigma} - E_{\mu\nu\sigma} = \alpha y^n_{,\mu}y_{n,\nu\sigma} \tag{7.34}$$

with an interchange of μ and ν on E one finds

$$\Gamma_{\mu\nu\sigma} + E_{\nu\mu\sigma} = \alpha y^n_{,\mu}y_{n,\nu\sigma} + g_{\mu\nu}(ln\alpha)_{,\sigma} \tag{7.35}$$

substitution into 7.23 results in

$$dA_\nu = A^\mu[\Gamma_{\mu\nu\sigma} + E_{\nu\mu\sigma}]dx^\sigma \tag{7.36}$$

or equivalently

$$dA_\nu = A^\mu[\Gamma_{\mu\nu\sigma} - E_{\mu\nu\sigma} + g_{\mu\nu}(ln\alpha)_{,\sigma}]dx^\sigma \tag{7.37}$$

to be compared with the standard result in GTR (which is obtained when $\alpha = 1$) of

$$dA_\nu = A^\mu\Gamma_{\mu\nu\sigma}dx^\sigma \tag{7.38}$$

If one then lets

$$\Gamma_{*\mu\nu\sigma} = \Gamma_{\mu\nu\sigma} + E_{\nu\mu\sigma} \tag{7.39}$$

then 7.36 can be written as

$$dA_\nu = A^\mu\Gamma_{*\mu\nu\sigma}dx^\sigma \tag{7.40}$$

It will become apparent that the standard GTR is modified by simply introducing the modified Christoffel symbol of 7.39 and that standard GTR is always recovered when $\alpha = 1$ in the modified theory.

7.0.28 The Change in Vector Length

Due to the local expansion, the above result for the change in a vector will now include a change in length of the vector $(d(g^{\mu\nu}A_\mu A_\nu))$ under parallel displacement which can now be determined as in the following.

$$d(g^{\mu\nu}A_\mu A_\nu) = g^{\mu\nu}[A_\mu dA_\nu + A_\nu dA_\mu] + A_\mu A_\nu g^{\mu\nu}_{,\sigma} dx^\sigma \tag{7.41}$$

with the usual use of raising/lowering indices and a change of summation indices in the last term on the RHS, one gets

$$d(g^{\mu\nu}A_\mu A_\nu) = A^\nu dA_\nu + A^\mu dA_\mu + A_\alpha A_\beta g^{\alpha\beta}_{,\sigma} dx^\sigma \tag{7.42}$$

now substitution of 7.37 yields

$$\begin{aligned} d(g^{\mu\nu}A_\mu A_\nu) =& [A^\nu A^\mu(\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma}) - A^\nu A^\mu(E_{\mu\nu\sigma} + E_{\nu\mu\sigma}) \\ & + 2A^\nu A_\nu(ln\alpha)_{,\sigma} + A_\alpha A_\beta g^{\alpha\beta}_{,\sigma}]dx^\sigma \end{aligned} \tag{7.43}$$

Now one can use the relationships

$$\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma} = g_{\mu\nu,\sigma} \tag{7.44}$$

and

$$E_{\mu\nu\sigma} + E_{\nu\mu\sigma} = g_{\mu\nu}(ln\alpha)_{,\sigma} \tag{7.45}$$

so substitution into 7.43 gives

$$\begin{aligned} d(g^{\mu\nu}A_\mu A_\nu) =& [A^\nu A^\mu g_{\mu\nu,\sigma} - A^\nu A_\nu (ln\alpha)_{,\sigma} \\ &+ 2A^\nu A_\nu (ln\alpha)_{,\sigma} + A_\alpha A_\beta g^{\alpha\beta}_{,\sigma}]dx^\sigma \end{aligned} \tag{7.46}$$

It is now useful to look at a relationship involving derivatives of the metric tensor, so if one takes

$$g^{\alpha\mu}_{,\sigma} g_{\mu\nu} + g^{\alpha\mu} g_{\mu\nu,\sigma} = (g^{\alpha\mu} g_{\mu\nu})_{,\sigma} = g^{\alpha}_{\nu,\sigma} = 0 \tag{7.47}$$

and then multiplies through by $g^{\beta\nu}$ one gets

$$g^{\alpha\beta}_{,\sigma} = -g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu,\sigma} \tag{7.48}$$

If one now takes this relationship and substitutes it into 7.46 and performs the arithmetic, the result is

$$d(g^{\mu\nu}A_\mu A_\nu) = A^\nu A_\nu (ln\alpha)_{,\sigma}dx^\sigma = A^\nu A_\nu d(ln\alpha) \tag{7.49}$$

so

$$d(A^\nu A_\nu) = A^\nu A_\nu d(ln\alpha) \tag{7.50}$$

showing a change in length under parallel displacement by a factor of $\frac{1}{\alpha}$, such that

$$d[(\frac{1}{\alpha})(A^\nu A_\nu)] = 0 \tag{7.51}$$

Now if one lets $A^\nu = dx^\nu$, then with reference to 7.10 one has

$$d[(\frac{1}{\alpha})(dx^\nu dx_\nu)] = d(\frac{ds'^2}{\alpha}) = 0 \tag{7.52}$$

and so $d(ds) = 0$ which gives an invariant distance for 7.9 as expected.

The change in a covariant vector has been determined by 7.40, the change in a contravariant vector now must be determined, not least because it is the expression for this change in a contravariant vector which is used in order to determine the equation for a time-like geodesic.

So following directly from 7.50 one has

$$d(A_\nu B^\nu) = A_\nu B^\nu d(ln\alpha) = A_\nu dB^\nu + B^\nu dA_\nu \tag{7.53}$$

substituting in 7.40 one gets

$$d(A_\nu B^\nu) = A_\nu dB^\nu + B^\nu A^\mu \Gamma_{*\mu\nu\sigma} dx^\sigma \tag{7.54}$$

then using $\Gamma_{*\mu\nu\sigma} = g_{\mu\alpha}\Gamma^{\alpha}_{*\nu\sigma}$ and a swap of summation indices (α and μ), one gets

$$d(A_\nu B^\nu) = A_\nu dB^\nu + B^\nu A_\mu \Gamma^{\mu}_{*\nu\sigma} dx^\sigma \tag{7.55}$$

Now rearranging, use of 7.50 and a swap of summation indices (ν and μ) and cancellation of the repeated A_ν term, one has

$$dB^\nu = B^\nu d(ln\alpha) - B^\mu \Gamma^{\nu}_{*\mu\sigma} dx^\sigma \tag{7.56}$$

If one now defines

$$\Gamma^{*\nu}_{\mu\sigma} = \Gamma^{\nu}_{*\mu\sigma} - g^{\nu}_{\mu}(ln\alpha)_{,\sigma} \tag{7.57}$$

multiplying through by $g_{\nu\alpha}$ one has

$$\Gamma^{*}_{\alpha\mu\sigma} = \Gamma_{*\alpha\mu\sigma} - g_{\alpha\mu}(ln\alpha)_{,\sigma} \tag{7.58}$$

now, as seen from 7.37 and 7.39 one has $\Gamma_{*\alpha\mu\sigma} = \Gamma_{\alpha\mu\sigma} - E_{\alpha\mu\sigma} + g_{\alpha\mu}(ln\alpha)_{,\sigma}$ so substituting this into 7.58 yields

$$\Gamma^{*}_{\alpha\mu\sigma} = \Gamma_{\alpha\mu\sigma} - E_{\alpha\mu\sigma} \tag{7.59}$$

for a consistent notation.

Now substituting 7.57 into 7.56 one gets

$$dB^{\nu} = B^{\nu}d(ln\alpha) - [B^{\mu}\Gamma^{*\nu}_{\mu\sigma} - B^{\nu}(ln\alpha)_{,\sigma}]dx^{\sigma} \tag{7.60}$$

therefore the change for a contravariant vector is given by

$$dB^{\nu} = -B^{\mu}\Gamma^{*\nu}_{\mu\sigma}dx^{\sigma} \tag{7.61}$$

It should now be apparent that the modification of the standard GTR by the use of the notion of a local expansion of space-time can simply be obtained by replacing the standard Christoffel symbols with the modified Christoffel symbols.

7.0.29 The Modified Time Like Geodesic, Curvature Tensor and Covariant Differentiation

The time-like geodesic equation of 5.4 is now

$$\frac{dv^{\mu}}{ds} = -\Gamma^{*\nu}_{\mu\sigma} v^{\nu} v^{\sigma} \tag{7.62}$$

and the curvature tensor of 3.55 is now modified to

$$R^{\beta}_{*\nu\rho\sigma} = \Gamma^{\beta}_{*\nu\sigma,\rho} - \Gamma^{\beta}_{*\nu\rho,\sigma} + \Gamma^{\alpha}_{*\nu\sigma}\Gamma^{\beta}_{*\alpha\rho} - \Gamma^{\alpha}_{*\nu\rho}\Gamma^{\beta}_{*\alpha\sigma} \tag{7.63}$$

Also the definition and operation of covariant differentiation from 3.45 and 3.46 is now

$$A^{\mu}(x+dx) - [A_{\mu}(x) + \Gamma^{\alpha}_{*\mu\nu} A_{\alpha} dx^{\nu}] = (A_{\mu,\nu} - \Gamma^{\alpha}_{*\mu\nu} A_{\alpha}) dx^{\nu} \tag{7.64}$$

The notation of 3.47 is now changed from a : to a ; to reflect the fact that one now has a modified covariant derivative, as such one has

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^{\alpha}_{*\mu\nu} A_{\alpha} \tag{7.65}$$

in which the modification has been constructed from the parallel displacement of a tensor and as such satisfies general covariance and now defines the new covariant derivative.

The d'Alembert equation of 5.32 is thus modified to

$$g^{\mu\nu}\Phi_{;\mu;\nu} = g^{\mu\nu}(\Phi_{,\mu\nu} - \Gamma^{\alpha}_{*\mu\nu}\Phi_{,\alpha}) = 0 \tag{7.66}$$

7.0.30 The Geodesic Acceleration

One can now consider the analogous Newtonian approximation which was dealt with in Chapter Five. The method and approximations are identical, the difference being is that there is now the extra expansion term 7.33 to consider, which has for each term, a derivative involving the expansion factor multiplied by the metric tensor.

So as well as the approximations leading to 5.10, 5.12, 5.13 and 5.14 one now has the following analogous results to be included in the new approximation analysis

$$(ln\alpha)_{,0} = 0 \tag{7.67}$$

$$E_{m0n} = 0 \tag{7.68}$$

leading to

$$\Gamma^{*}_{m0n} = \Gamma_{*m0n} = 0 \tag{7.69}$$

$$\Gamma^{*m}_{0n} = \Gamma^{m}_{*0n} = 0 \tag{7.70}$$

So starting the new analysis, from 7.10 one has

$$\alpha ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{7.71}$$

therefore following the same definition and procedure as in Chapter Five one has

$$\alpha = g_{\mu\nu} v^\mu v^\nu \tag{7.72}$$

and using exactly the same approximations as in Chapter Five this leads to

$$g_{00}(v^0)^2 = \alpha \tag{7.73}$$

which is to be compared with 5.17.

One now moves on to the geodesic equation to find the new potential analogous to that of Chapter Five.

The modified time-like geodesic equation is that of 7.62 which is

$$\frac{dv^\mu}{ds} = -\Gamma^{*\nu}_{\mu\sigma} v^\nu v^\sigma \tag{7.74}$$

again using the slow moving approximation, dropping quadratics and also the use of 7.70 leads to

$$\begin{aligned}
\frac{dv^m}{ds} &= -\Gamma^{*m}_{\nu\sigma} v^\nu v^\sigma \\
&= -\Gamma^{*m}_{00} (v^0)^2 \\
&= -g^{mn} \Gamma^*_{n00} (v^0)^2
\end{aligned} \tag{7.75}$$

Now from 7.59 one has $\Gamma^*_{n00} = \Gamma_{n00} - E_{n00}$, so in the usual manner from Chapter Five using the static field approximation, one has

$$\begin{aligned}
\Gamma_{n00} &= \frac{1}{2}[g_{n0,0} + g_{0n,0} - g_{00,n}] \\
&= -\frac{1}{2} g_{00,n}
\end{aligned} \tag{7.76}$$

and also using the static field approximation of 7.67 one also has

$$\begin{aligned}
E_{n00} &= \frac{1}{2}[g_{n0}(ln\alpha)_{,0} + g_{0n}(ln\alpha)_{,0} - g_{00}(ln\alpha)_{,n}] \\
&= -\frac{1}{2} g_{oo}(ln\alpha)_{,n}
\end{aligned} \tag{7.77}$$

so 7.75 now becomes

$$\frac{dv^m}{ds} = \frac{1}{2} g^{mn}[g_{00,n}(v^0)^2 - g_{00}(ln\alpha)_{,n}(v^0)^2] \tag{7.78}$$

Now from 7.73 one has $(v^0)^2 = \frac{\alpha}{g_{00}}$, so

$$\begin{aligned} \frac{dv^m}{ds} &= \frac{1}{2}\alpha g^{mn}[\frac{g_{00,n}}{g_{00}} - (ln\alpha)_{,n}] \\ &= \frac{1}{2}\alpha g^{mn}(ln\frac{g_{00}}{\alpha})_{,n} \end{aligned} \tag{7.79}$$

This can now be written

$$\frac{dv^m}{ds} = \alpha g^{mn}\phi_{,n} \tag{7.80}$$

where the new potential ϕ is now seen to be

$$\phi = ln\sqrt{\frac{g_{00}}{\alpha}} \approx \sqrt{\frac{g_{00}}{\alpha}} - 1 \approx \frac{1}{2}(\frac{g_{00}}{\alpha} - 1) \tag{7.81}$$

So, to check for consistency, when $\alpha = 1$ then the weak field approximation is such that $g_{00} = 1 + 2\Phi$, substitution into 7.79 (also remembering the diagonal elements of $g^{mn} = -1$) yields

$$\begin{aligned} \frac{dv^m}{ds} &= -(ln\sqrt{1+2\Phi})_{,n} \\ &= -[ln(1+\Phi)]_{,n} \\ &= -\Phi_{,n} \end{aligned} \tag{7.82}$$

This is then consistent with 5.37 and 5.38 since when $\alpha = 1$ the standard Newtonian approximation of Einstein's law derived in Chapter Five holds.

The modification to the gravitational potential now depends on α (the local expansion of space time), which is tied in to the gravitational force, so, the gravitational potential is now seen to be comprised of two 'parts' - the curvature (as per Einstein), and the local expansion. Both parts are present throughout. However the curvature dominates the expansion for small local systems (e.g solar system), to the extent that the expansion is negligible for gravitational effects, therefore leaving standard GTR/Newton unaltered. However for large systems/large distances (i.e. galaxies), it is the expansion term which dominates leading to an adjusted gravitational effect.

The metric tensor and its degrees of freedom are thus unchanged for the General Relativistic view of the solar system. The extra degree of freedom for the gravitational description of galactic systems due to the parameter α is explained physically by the local expansion of space time. As such there is no contradiction or 'spare part'.

7.0.31 The Modified Einstein Field Equations and the MOND Approximation

The very same analysis must now be completed using the new modified law.

The general modified law for empty space is such that

$$R_{*\mu\nu} - \frac{1}{2}g_{\mu\nu}R_* = 0 \tag{7.83}$$

Now recalling 5.50, which is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi\rho v^\mu v^\nu \tag{7.84}$$

where the RHS has the inclusion of matter into Einstein's law. Since one now also needs a modification of this law, then the expansion factor must be incorporated into the RHS as well as the LHS. This can be achieved by defining a velocity v_*^μ such that

$$v_*^\mu = \frac{dx^\mu}{ds'} = \frac{1}{\sqrt{\alpha}}v^\mu \tag{7.85}$$

and since one has $\alpha ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, therefore

$$\alpha = g_{\mu\nu}v^\mu v^\nu \tag{7.86}$$

and so

$$g_{\mu\nu}v_*^\mu v_*^\nu = 1 \tag{7.87}$$

which means that $v_{*\mu}v^{\nu}_{*;\sigma} = 0$. Since the modified velocity is just that of 7.85, then the same conservation analysis of Chapter Five will ensue. The condition for the conservation of matter is such that

$$(\rho v_*^{\mu})_{;\mu} = 0 \tag{7.88}$$

so for the material energy tensor $T^{\mu\nu}$ one has

$$T^{\mu\sigma}_{;\mu} = (\rho v_*^{\mu} v_*^{\nu})_{;\mu} = 0 \tag{7.89}$$

The new modified law with the inclusion of matter(in lowered index form)is thus

$$R_{*\mu\nu} - \frac{1}{2} g_{\mu\nu} R_* = -8\pi\rho v_{*\mu} v_{*\nu} \tag{7.90}$$

again it should be noted here that when $\alpha = 1$ then $R_* = R$ and $v_* = v$ and so standard GTR is recovered.

Now, the $R_{*\mu\nu}$ must be calculated, and as in standard GTR, in the weak field approximation the quadratic terms may be neglected, and with use of 7.39 one has

$$
\begin{aligned}
R_{*\mu\nu} &= \Gamma^{\alpha}_{*\mu\alpha,\nu} - \Gamma^{\alpha}_{*\mu\nu,\alpha} \\
&= g^{\alpha\beta}[\Gamma_{*\beta\mu\alpha,\nu} - \Gamma_{*\beta\mu\nu,\alpha}] \\
&= g^{\alpha\beta}[\Gamma_{\beta\mu\alpha,\nu} + E_{\mu\beta\alpha,\nu} - \Gamma_{\beta\mu\nu,\alpha} - E_{\mu\beta\nu,\alpha}] \\
&= g^{\alpha\beta}[\Gamma_{\beta\mu\alpha,\nu} - \Gamma_{\beta\mu\nu,\alpha}] + g^{\alpha\beta}[E_{\mu\beta\alpha,\nu} - E_{\mu\beta\nu,\alpha}]
\end{aligned} \tag{7.91}
$$

For completeness, using 3.38 and 7.33 the two terms on the RHS of 7.91 may be calculated separately, so one has

$$
\begin{aligned}
g^{\alpha\beta}[\Gamma_{\beta\mu\alpha,\nu} - \Gamma_{\beta\mu\nu,\alpha}] = g^{\alpha\beta}[&\frac{1}{2}(g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} + g_{\beta\alpha,\mu\nu}) \\
&- \frac{1}{2}(g_{\beta\mu,\nu\alpha} - g_{\nu\mu,\beta\alpha} + g_{\beta\nu,\mu\alpha})]
\end{aligned} \tag{7.92}
$$

and therefore

$$
g^{\alpha\beta}[\Gamma_{\beta\mu\alpha,\nu} - \Gamma_{\beta\mu\nu,\alpha}] = \frac{1}{2}g^{\alpha\beta}[g_{\beta\alpha,\mu\nu} - g_{\alpha\mu,\beta\nu} - g_{\nu\mu,\beta\alpha} - g_{\beta\nu,\mu\alpha}] \tag{7.93}
$$

and for the second term one has

$$g^{\alpha\beta}[E_{\mu\beta\alpha,\nu} - E_{\mu\beta\nu,\alpha}] = \frac{1}{2}g^{\alpha\beta}[(g_{\mu\beta}(ln\alpha)_{,\alpha})_{,\nu} - (g_{\alpha\beta}(ln\alpha)_{,\mu})_{,\nu} + (g_{\mu\alpha}(ln\alpha)_{,\beta})_{,\nu}]$$
$$- \frac{1}{2}g^{\alpha\beta}[(g_{\mu\beta}(ln\alpha)_{,\nu})_{,\alpha} - (g_{\nu\beta}(ln\alpha)_{,\mu})_{,\alpha} + (g_{\mu\nu}(ln\alpha)_{,\beta})_{,\alpha}] \tag{7.94}$$

which with cancellations and differentiation performed, gives

$$\begin{aligned} g^{\alpha\beta}[E_{\mu\beta\alpha,\nu} - E_{\mu\beta\nu,\alpha}] = \frac{1}{2}g^{\alpha\beta}[&-g_{\alpha\beta}(ln\alpha)_{,\mu\nu} - (ln\alpha)_{,\mu}g_{\alpha\beta,\nu} \\ &+ g_{\mu\alpha}(ln\alpha)_{,\beta\nu} + (ln\alpha)_{,\beta}g_{\mu\alpha,\nu} \\ &- g_{\nu\beta}(ln\alpha)_{,\mu\alpha} + (ln\alpha)_{,\mu}g_{\nu\beta,\alpha} \\ &+ g_{\mu\nu}(ln\alpha)_{,\beta\alpha} + (ln\alpha)_{,\beta}g_{\mu\nu,\alpha}] \end{aligned} \tag{7.95}$$

Now returning to the new modification of Einstein's law of gravity of 7.90, which is

$$R_{*\mu\nu} - \frac{1}{2}g_{\mu\nu}R_* = -8\pi\rho v_{*\mu}v_{*\nu} \tag{7.96}$$

and recalling the definition of v^{μ}_{*} from 7.85, one can multiply through by α and obtain

$$\alpha[R_{*\mu\nu} - \frac{1}{2}g_{\mu\nu}R_*] = -8\pi\rho v_{\mu}v_{\nu} \tag{7.97}$$

now with a rearrangement equivalent to that of 5.54, the condition of 5.55, a substitution of $R_{*\mu\nu}$ from 7.91 and the approximation condition $\mu = \nu = 0$ (and $g_{00} \approx \alpha$), then 7.97 becomes

$$\alpha g^{\alpha\beta}[\Gamma_{\beta 0\alpha,0} - \Gamma_{\beta 00,\alpha}] + \alpha g^{\alpha\beta}[E_{0\beta\alpha,0} - E_{0\beta 0,\alpha}] = -4\pi\rho v_0 v_0 \tag{7.98}$$

Now the static field approximations of $g_{\mu\nu,0} = 0$ and $(ln\alpha)_{,0} = 0$ need to be applied to the Christoffel and Expansion symbols, it is clear that for $\Gamma_{\beta 0\alpha,0}$ and $E_{0\beta\alpha,0}$ one will get

$$\Gamma_{\beta 0\alpha,0} = 0 \tag{7.99}$$

and

$$E_{0\beta\alpha,0} = 0 \tag{7.100}$$

calculation of the other two terms gives

$$\begin{aligned}\Gamma_{\beta 00,\alpha} &= \frac{1}{2}[g_{\beta 0,0\alpha} - g_{00,\beta\alpha} + g_{\beta 0,0\alpha}] \\ &= -\frac{1}{2}g_{00,\beta\alpha}\end{aligned} \tag{7.101}$$

and

$$
\begin{aligned}
E_{0\beta 0,\alpha} &= \frac{1}{2}[(g_{0\beta}(ln\alpha)_{,0})_{,\alpha} - (g_{0\beta}(ln\alpha)_{,0})_{,\alpha} + (g_{00}(ln\alpha)_{,\beta})_{,\alpha}] \\
&= \frac{1}{2}[g_{00}(ln\alpha)_{,\beta}]_{,\alpha}
\end{aligned} \tag{7.102}
$$

so now substitution of 7.101 and 7.102 into 7.98 yields

$$
\frac{\alpha}{2} g^{\alpha\beta}[g_{00,\beta\alpha} - (g_{00}(ln\alpha)_{,\beta})_{,\alpha}] = -4\pi\rho v_0 v_0 \tag{7.103}
$$

and one therefore has

$$
\alpha g^{mn}[g_{00,nm} - (g_{00}(ln\alpha)_{,n})_{,m}] = -8\pi\rho v_0 v_0 \tag{7.104}
$$

Now for a weak field one has

$$
y^n_{,\mu} y_{n,\nu} \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7.105}
$$

and so from 7.8 one has

$$g_{\mu\nu} = \alpha y^{n}_{,\mu} y_{n,\nu} \approx \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix} \tag{7.106}$$

and therefore

$$g^{\mu\nu} \approx \begin{pmatrix} {}^{1}/_{\alpha} & 0 & 0 & 0 \\ 0 & {}^{-1}/_{\alpha} & 0 & 0 \\ 0 & 0 & {}^{-1}/_{\alpha} & 0 \\ 0 & 0 & 0 & {}^{-1}/_{\alpha} \end{pmatrix} \tag{7.107}$$

Now from 7.73 and using $g^{00} = \frac{1}{g_{00}}$ for a static and weak field one has $g^{00} v_o v_0 =^1 /_{\alpha}$ and from the above this means $v_0 = 1$ also from the above $g^{mn} = \frac{1}{\alpha}$, so substitution into 7.104 gives

$$g_{00,mm} - [g_{00}(ln\alpha)_{,m})_{,m}] = 4\pi\rho \tag{7.108}$$

Now from 7.81 one has

$$\phi_{,n} = \frac{1}{2}[ln(g_{00})_{,n} - (ln\alpha)_{,n}] \tag{7.109}$$

so rearranging, one has

$$ln(g_{00})_{,n} - (ln\alpha)_{,n} = 2\phi_{,n} \tag{7.110}$$

therefore

$$\frac{g_{00,n}}{g_{00}} - (ln\alpha)_{,n} = 2\phi_{,n} \tag{7.111}$$

multiplying through by g_{00} and rearranging gives

$$g_{00}(ln\alpha)_{,n} = g_{00,n} - 2g_{00}\phi_{,n} \tag{7.112}$$

so substituting 7.112 into 7.108 gives

$$g_{00,mm} - [g_{00,m} - 2g_{00}\phi_{,m}]_{,m} = 8\pi\rho \tag{7.113}$$

therefore

$$[2g_{00}\phi_{,m}]_{,m} = 8\pi\rho \tag{7.114}$$

and finally one has

$$(g_{00}\phi_{,m})_{,m} = 4\pi\rho \tag{7.115}$$

which can be compared directly with 6.5, providing g_{00} can be identified as the MOND interpolation function μ.

Now since from rearranging 7.81 one has

$$\frac{g_{00}}{\alpha} = 1 + 2\Phi \tag{7.116}$$

then it is clear that although $\frac{g_{00}}{\alpha}$ must be approximately unity, g_{00} itself is free to vary as the MOND interpolation function μ.

7.0.32 Point Source and General Solution

With 6.4 in mind, one now has for a point source, Newton's second law written as $F = mg_{00}a$. Where m is the point mass, a is the acceleration and F is the force. As seen from 7.116 g_{00} is free to vary as the MOND interpolation function. Also, as one can see from 5.17, there is no restriction on g_{00} in standard GTR.

Rearranging (7.81) and then substituting the result $g_{00} = \alpha e^{2\phi}$ into (7.115) and performing integration gives

$$\alpha e^{2\phi} = \frac{D}{\left(r^2|\tilde{\bigtriangledown}\phi|\right)} \tag{7.117}$$

for point source of mass M, $\rho = M\delta(r)$, where D is an integration constant, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\delta(r)$ is the Dirac delta function.

If one then matches this solution to the observed flattening of galaxys' rotation curves as discussed in chapter 6, then these observations impose that $|\tilde{\bigtriangledown}\phi| \to D/r^2$ when $|\tilde{\bigtriangledown}\phi| \gg a_0$ and that $|\tilde{\bigtriangledown}\phi| \to \sqrt{a_0 D}/r$ when $|\tilde{\bigtriangledown}\phi| \ll a_0$, where a_0 is the acceleration parameter of MOND theory.

Thus a consistent solution for the potential is derived to be

$$\phi = -M/r + \sqrt{a_0 M}\ln r, \tag{7.118}$$

where D has been identified as the point source mass M.

This derivation is based on observation and allows an interpretation of the rate of expansion, which suggests a physical context and thus an alternative derivation (see later).

The first term in 7.118 is obviously the Newtonian potential due to the curvature ϕ^{NEWT} as derived in Chapter Five, and the second term is the MOND potential (from Chapter Six) which is due to local expansions ϕ^{MOND}, see fig. 7.1.

From 7.118 one has

$$\phi_{,m} = [M/r^2 + \frac{\sqrt{a_0 M}}{r}]\vec{r} \tag{7.119}$$

Now since one has $(g_{00}\phi_{,m})_{,m} = 4\pi\rho$ (7.115) this means that

$$g_{00}\phi_{,m} = (M/r^2)\vec{r} \tag{7.120}$$

so rearranging 7.120 and substituting in for 7.119 one has

$$g_{00} = \frac{M/r^2}{M/r^2 + \sqrt{a_0 M}/r}, \tag{7.121}$$

and substituting 7.118 into 7.116 one has

$$g_{00}/\alpha = 1 - 2M/r + 2\sqrt{a_o M}\ln r \tag{7.122}$$

as such two limits can now be considered:

For small r such that the curvature term M/r^2 dominates the expansion term $\sqrt{a_0 M}/r$, then this equates to a dominant solution of the Newtonian potential ϕ^{NEWT} where the accelerations are such that $|\phi^N{}_{,m}|/a_0 >> 1$.

This means that (7.118) becomes

$$\phi^{NEWT} = -M/r \tag{7.123}$$

and therefore

$$\phi^{NEWT}{}_{,m} = (M/r^2)\vec{r} \tag{7.124}$$

and also (7.121) therefore becomes

$$g_{00} = 1 - \sqrt{\frac{a_0}{M}} r \approx 1 \tag{7.125}$$

and so from 7.122 one has

$$g_{00}/\alpha = 1 - 2M/r \tag{7.126}$$

thus the Newtonian point source potential is recovered.

For large r such that the expansion term $\sqrt{a_0 M}/r$ dominates the curvature term M/r^2, then this equates to a dominant solution of the MOND potential ϕ^{MOND} where the accelerations are such that $|\phi^M{}_{,m}|/a_0 << 1$.

Thus (7.118) becomes

$$\phi^{MOND} = \sqrt{a_0 M} \ln r \tag{7.127}$$

and therefore

$$\phi^{MOND}{}_{,m} = (\sqrt{a_0 M}/r)\vec{r} \tag{7.128}$$

and (7.121) becomes

$$g_{00} = \sqrt{\frac{M}{a_0}} \frac{1}{r} \tag{7.129}$$

So substituting 7.127 and 7.129 into (7.115), gives

$$[\sqrt{\frac{M}{a_0}}\frac{1}{r}(\sqrt{a_0 M}\ln r)_{,m}]_{,m} = [\frac{M}{r^2}\vec{r}]_{,m} = 4\pi M\delta(x) = 4\pi\rho \tag{7.130}$$

as expected.

Also if limits are introduced directly into (7.117) such that for the Newtonian case as $r \to 0$, $\alpha = 1$ and $2\phi \ll 1$, this gives

$$r^2\,(1+2\phi)\,\tilde{\bigtriangledown}\phi = M. \tag{7.131}$$

After integration and $|\phi^2| \ll |\phi|$ yields $\phi = -M/r$ as expected.

In the MOND limit $|\phi^2| \gg |\phi^3|$ and $\alpha = \alpha(r)$. After integration this gives $\phi(r) \propto \sqrt{a_0 M}\ln r$, where

$$\alpha(r) = \frac{1}{2a_0 r\ln r}. \tag{7.132}$$

Interestingly, the $1/r\ln r$ dependence for α (the space-time expansion) is identical to the large r radial velocity of a spherical shock wave [56] [28]. So, if one assumed this physical origin for expansion one can directly derive the second term in (7.118) without fitting MOND characteristics to the solution.

It is noted that the factor g_{00} is approximately unity in the Newtonian approximation, meaning that (7.115) is linear and so a system of point sources can be considered as a summation of separate point source solutions. However, in the MOND approximation, g_{00} is a varying function, and so (7.115) is non-linear and cannot be broken down in this way. Furthermore, the mass term on the right hand side of (7.115) is split into a factor $\sqrt{M}$ with the potential and a factor $\sqrt{M}$ with g_{00}. So, the momentum equation of Newton's second law only makes sense if it is modified to include the factor g_{00}. Furthermore, because of the nonlinearity this factor g_{00} can only be calculated once the complete system is known.

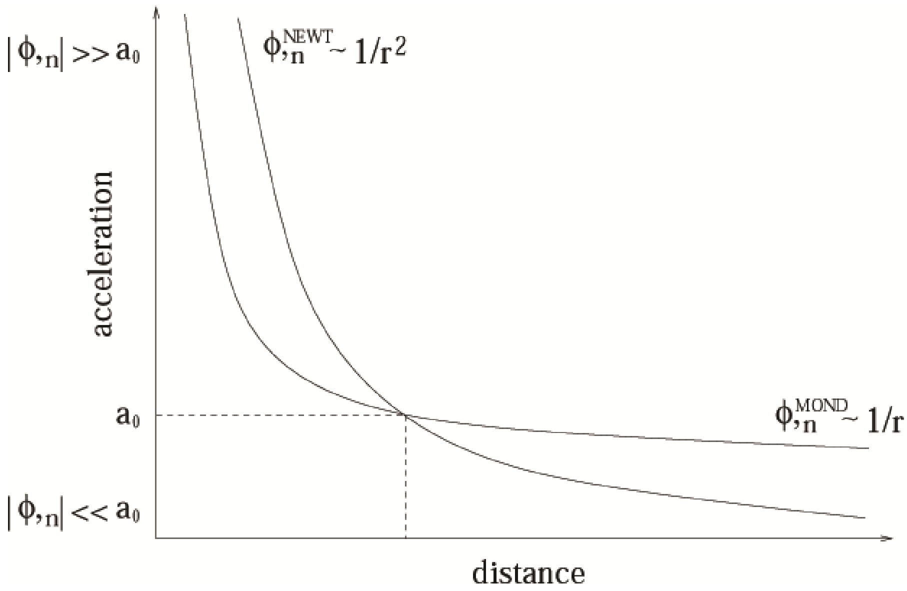

FIGURE 7.1: The change in acceleration with distance

The point source solution suggests a general solution given by

$$
\begin{aligned}
\phi &= \phi^{NEWT} + \phi^{MOND} \\
g_{00} &= \left| \frac{\nabla \phi^{NEWT}}{\nabla \phi^{NEWT} + \nabla \phi^{MOND}} \right| \\
\alpha &= \frac{1 - 2\phi^{NEWT} - 2\phi^{MOND}}{1 + |\nabla \phi^{MOND} / \nabla \phi^{NEWT}|} \\
g_{00}/\alpha &= 1 + 2\phi^{NEWT} + 2\phi^{MOND}
\end{aligned}
$$

where ∇ is the differential operator $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ for Cartesian co-ordinate system vector representation (x_1, x_2, x_3). ϕ^{NEWT} and ϕ^{MOND} are connected in the sense that they can be seen as limiting values of the same general potential ϕ, such that the first is the limit of small relative radius for solar systems, and the second is the limit of large relative radius for galaxies. So this choice of ϕ has a certain degree of physical justification in that it gives the expected physics in these limits. The two limits are then as follows.

When $|\phi,_n|/a_0 >> 1$, then curvature dominates so $|\nabla\phi^{NEWT}| >> |\nabla\phi^{MOND}|$, and

$$\begin{aligned} \phi &= \phi^{NEWT} \\ g_{00} &= 1 \\ \alpha &= 1 - 2\phi^{NEWT} \\ g_{00}/\alpha &= 1 + 2\phi^{NEWT}, \end{aligned}$$

and so $\phi^{NEWT},_{mm} = 4\pi\rho$, and the Newtonian gravitational representation is recovered. Such accelerations feature in solar system dynamics.

However, when $|\phi,_n|/a_0 << 1$, then expansion dominates $|\nabla\phi^{MOND}| >> |\nabla\phi^{NEWT}|$, and

$$
\begin{aligned}
\phi &= \phi^{MOND} \\
g_{00} &= \left|\frac{\nabla\phi^{MOND}}{a_0}\right| \\
\alpha &= \left|\frac{\nabla\phi^{MOND}}{a_0}\right|(1 - 2\phi^{MOND}) \\
g_{00}/\alpha &= 1 + 2\phi^{MOND},
\end{aligned}
$$

and so (7.115) becomes

$$
(g_{00}\phi_{,m})_{,m} = (\frac{|\nabla\phi^{MOND}|}{a_0}\phi^{MOND}{}_{,m})_{,m} = 4\pi\rho,
$$

which is the MOND representation for the potential acceleration, and as seen in Chapter Six, it is these accelerations that feature in the motions of galaxies.

Thus the analysis within this chapter would appear to show that, by introducing the physical assumption of a local space-time expansion, due to the presence of mass, into standard GTR one can explain the observed rotational motions of the galaxies and yet at the same time leave the standard GTR unaltered to explain the observed motions of the solar system without assuming any Dark Matter.

Chapter 8

CONCLUSION

8.1 Main Section 1

From the analysis presented in the previous chapters it would seem that the introduction, into standard GTR, of the notion of a local expansion of space-time and the effects this has on the metric, can be seen to result in the observed MONDian phenomenology and thus eliminating the need for any Dark Matter.

The main problem for the MOND approach has been the lack of a relativistic backing and also the lack of a physical basis (why should anything like MOND happen?). The assumption that there is local space-time expansion (which stems directly from the non-Riemmanian geometry, or vice versa) may provide the much needed physical underpinning for a relativistic MOND theory.

This physical mechanism may also help to explain the numerical coincidence that $a_0 \approx cH_0$ (where c is the velocity of light and H_0 is the Hubble constant), which suggests a connection of MOND to the expansion of the Universe.

At present the choice of g_{00} for the interpolation function is not backed by a fully compelling natural derivation that says it must be so. However the non-Riemmanian geometry resulting from the introduction of local space-time expansion may actually be a necessity. This may possibly be shown to originate from the rotating disk scenario in Special Relativity where large distances are concerned. If this can be shown to be the case then it might be expected that the g_{00} would naturally become the MOND interpolation function.

In modern cosmology there is also the need to include the mysterious property of Dark Energy (DE). DE was originally proposed as the major contribution to the Universe's energy density in order to account for a flat Euclidean universe which was favoured by inflationary models [15]. The DE proposal was then further refined in the late 1990's when the Universe was observed to be expanding at an *accelerating* rate [43]. There are curious coincidences of scales between the DM and DE sectors [38], showing that these two areas may not be physically independent and as such the assumption of a local space-time expansion may possibly rid the need for DE also. This may possibly be achieved by introducing the phenomena of local space-time expansion into into the standard analysis for the gravitational red shift.

Early Universe cosmology concerns a regime where the pressure term in the new energy momentum tensor is no longer negligible. This situation requires analysis which may give rise to explanations involving the evolution of Galaxies and is intended to be explored in further work.

Finally, there is the phenomena of galactic lensing [40], which from observation requires an extra deflection of light in addition to that given by standard GTR. Although the work presented within this thesis assumes a conformal transform when dealing with the electromagnetic field, which would normally imply that there was no extra deflection, the viewpoint of a local space-time expansion (that can only be detected in large systems/distances) may possibly be seen to give the extra deflection.

Appendix A

PUBLICATIONS

14. Selected News Articles: (i) TF Hodgkinson and GS McDonald, New theory of general relativity casts doubt on dark matter, The Hindu, 5 August, 2013 [Readership of around 2,258,000]

(ii) TF Hodgkinson and GS McDonald, New theory of general relativity casts doubt on dark matter, newsco.me, Science News, 8 August, 2013

(iii) EA Chadwick, TF Hodgkinson and GS McDonald, Salford University scientists modify Einstein's equations and cast doubt on dark matter, The Manchester Gazette, Education, 23 July, 2013

(iv) Z de Belder, EA Chadwick, TF Hodgkinson and GS McDonald, Did Einstein get something wrong? Salford uni scientists reveal theory that casts doubt on dark matter, MancunianMatters, News, 23 July, 2013

(v) TF Hodgkinson and GS McDonald, New theory of general relativity casts doubt on dark matter, SBS World News, Australia, 6 August, 2013

(vi) TF Hodgkinson and GS McDonald, New theory of general relativity casts doubt on dark matter, The Epoch Times, Science & Environment, 16-22 August, 2013

13. TF Hodgkinson and GS McDonald, New theory of general relativity casts doubt on dark matter, THE CONVERSATION, 5 August, 2013 [Invited-Authored Article, Readership of over 1,000,000 per month]

12. EA Chadwick, TF Hodgkinson and GS McDonald, Salford University scientists modify Einstein's equations and cast doubt on dark matter, News Release, 23 July, 2013

11. EA Chadwick, TF Hodgkinson and GS McDonald, Gravitational theoretical development supporting MOND, PHYSICAL REVIEW D v88, iss 2, art no 024036, 2013 [Impact Factor: 4.691]

10. JM Christian, GS McDonald, TF Hodgkinson and P Chamorro-Posada, Wave envelopes with second-order spatiotemporal dispersion. I. Bright Kerr solitons and cnoidal waves, PHYSICAL REVIEW A, v86, iss 2, art no 023838, 2012 [Impact Factor: 3.042]

9. JM Christian, GS McDonald, TF Hodgkinson and P Chamorro-Posada, Wave envelopes with second-order spatiotemporal dispersion. II. Modulational instabilities and dark Kerr solitons, PHYSICAL REVIEW A, v86, iss 2, art no 023839, 2012 [Impact Factor: 3.042]

8. GS McDonald, JM Christian, TF Hodgkinson, R Cowey, & G Lancaster, Spatiotemporal pulse propagation: connections to special relativity theory, Science &

Technology College Research Showcase Day, University of Salford, UK, 20 June, 2012.

7. JM Christian, GS McDonald, TF Hodgkinson and P Chamorro-Posada, Spatiotemporal Dispersion and Wave Envelopes with Relativistic and Pseudo-Relativistic Characteristics, PHYSICAL REVIEW LETTERS v108 no 3, Art. No. 034101, 2012 [Impact Factor: 7.943]

6. MJ Lundie, G Lancaster, TF Hodgkinson, JM Christian, GS McDonald and P Chamorro-Posada, Pulses with Relativistic and Pseudo-Relativistic Aspirations, Waves 2011 - 10th International Conference on Mathematical and Numerical Aspects of Waves, 25-29 July, Vancouver, Canada, 2011

5. G Lancaster, MJ Lundie, TF Hodgkinson, JM Christian and GS McDonald, Universality in Relativistic Aspirations of Helmholtz Soliton Pulses, CLEO Europe - EQEC 2011, 22-26 May, Munich, Germany, 2011

4. GS McDonald, JM Christian and TF Hodgkinson, Optical soliton pulses with relativistic characteristics, IEEE Abstract, 5th International Conference on Advanced Optoelectronics & Lasers, CAOL 2010, Sevastopol, Crimea, Ukraine, 10-14 Sept 2010 [Invited Conference Paper & Plenary Talk]

3. JM Christian, GS McDonald, TF Hodgkinson and P Chamorro-Posada, Optical pulses with spatial dispersion - exact solitons & relativity, National Photonics Conference, Photon 10, Southampton, UK, 23-26 Aug 2010

2. GS McDonald, JM Christian and TF Hodgkinson, Relativistic Effects in Non-Relativistic Systems, SEE Celebration of Research, meeting abstract & presentation, University of Salford, UK, 30 June 2010

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